

# SPECTRAL MULTIPLIER THEOREMS VIA $H^\infty$ CALCULUS AND $R$ -BOUNDS

CHRISTOPH KRIEGLER AND LUTZ WEIS

**ABSTRACT.** We prove spectral multiplier theorems for Hörmander classes  $\mathcal{H}_p^\alpha$  for 0-sectorial operators  $A$  on Banach spaces assuming a bounded  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \pi)$  and norm and certain  $R$ -bounds on one of the following families of operators: the semigroup  $e^{-zA}$  on  $\mathbb{C}_+$ , the wave operators  $e^{isA}$  for  $s \in \mathbb{R}$ , the resolvent  $(\lambda - A)^{-1}$  on  $\mathbb{C} \setminus \mathbb{R}$ , the imaginary powers  $A^{it}$  for  $t \in \mathbb{R}$  or the Bochner-Riesz means  $(1 - A/u)_+^\alpha$  for  $u > 0$ . In contrast to the existing literature we neither assume that  $A$  operates on an  $L^p$  scale nor that  $A$  is self-adjoint on a Hilbert space. Furthermore, we replace (generalized) Gaussian or Poisson bounds and maximal estimates by the weaker notion of  $R$ -bounds, which allow for a unified approach to spectral multiplier theorems in a more general setting. In this setting our results are close to being optimal. Moreover, we can give a characterization of the ( $R$ -bounded)  $\mathcal{H}_1^\alpha$  calculus in terms of  $R$ -boundedness of Bochner-Riesz means.

## 1. INTRODUCTION

Classical spectral multiplier theorems go back to Mihlin and Hörmander [31] who proved that “Fourier multiplier operators”

$$(1.1) \quad u \mapsto f(-\Delta)u(\cdot) = \mathcal{F}^{-1}[f(|\xi|^2)\hat{u}(\xi)](\cdot)$$

are bounded on  $L^q(\mathbb{R}^d)$ ,  $1 < q < \infty$  if  $f$  satisfies a smoothness condition such as

$$(1.2) \quad \sup_{R>0} \int_{R/2}^{2R} \left| t^k f^{(k)}(t) \right|^2 \frac{dt}{t} < \infty \quad (k = 0, 1, \dots, \alpha), \quad \alpha = \lceil d/2 \rceil.$$

Let us denote by  $\mathcal{H}_2^\alpha$  the “Hörmander class” of all functions  $f : \mathbb{R}_+ \rightarrow \mathbb{C}$  satisfying (1.2). Then the Fourier multiplier theorem of Hörmander says that the mapping

$$f \in \mathcal{H}_2^\alpha \mapsto f(-\Delta) \in B(L^q(\mathbb{R}^d))$$

defines an algebra homomorphism. There is by now a large literature extending such spectral multiplier results to Laplace-type operators (including elliptic and Schrödinger operators) on  $L^q$  spaces on manifolds, Lie groups and graphs [48, 1, 50, 13, 49, 20, 11, 21, 3, 22, 10], see also [52] and the references therein. Typical consequences of spectral multiplier theorems for  $f \in \mathcal{H}_2^\alpha$  and an operator  $A$  on  $L^q$  are norm estimates for important operator families attached to  $A$ , e.g. the norm

---

*Date:* December 14, 2016.

*2010 Mathematics Subject Classification.* 42A45, 47A60, 47B40, 47D03.

*Key words and phrases.* Functional calculus, Hörmander Type Spectral Multiplier Theorems.

The first named author acknowledges financial support from the Franco-German University (DFH-UFA) and the Karlsruhe House of Young Scientists (KHYS). The second named author acknowledges support from CRC 1173, DFG (Deutsche Forschungsgemeinschaft).

boundedness of the sets

$$\begin{aligned}
& (S)_\alpha & \left\{ \left( \frac{\pi}{2} - |\theta| \right)^\alpha \exp(-te^{i\theta}A) : t > 0, \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\} \\
& (W)_\alpha & \left\{ (1 + |s|A)^{-\alpha} \exp(isA) : s \in \mathbb{R} \right\} & \text{Wave operators} \\
& (\text{BIP})_\alpha & \left\{ (1 + |t|)^{-(\alpha+\epsilon)} A^{it} : t \in \mathbb{R} \right\} & \text{Bounded imaginary powers} \\
& (\text{BR})_\alpha & \left\{ (1 - A/u)_+^{\alpha+\epsilon-\frac{1}{2}} : u > 0 \right\} & \text{Bochner-Riesz means}
\end{aligned}$$

for any  $\epsilon > 0$ . Conversely, a common strategy to establish spectral multiplier theorems is to start from one of these families and estimate the corresponding representation formulas

$$\begin{aligned}
(1.3) \quad f(A) &= \frac{1}{2\pi} \int_{\mathbb{R}} \check{f}(t) e^{-itA} dt, & \check{f} \text{ the inverse Fourier-transform} \\
f(A) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathfrak{M}(f)(s) A^{is} ds, & \mathfrak{M}(f) \text{ the Mellin-transform} \\
f(A) &= \frac{(-1)^{[\alpha]}}{\Gamma(\alpha)} \int_0^\infty f^{(\alpha)}(u) (u - A)_+^{\alpha-1} u du, & f^{(\alpha)} \text{ the (fractional) derivative}
\end{aligned}$$

to obtain bounded operators  $f(A)$  on  $L^q(U)$  for all  $f \in \mathcal{H}_2^\alpha$ .

Usually, in a first step, one estimates one of the equations in (1.3) for a function  $f \in C_c^\infty(0, \infty)$  by using (generalized) Gaussian or Poisson kernel bounds or maximal estimates for the corresponding operator family in  $((S)_\alpha)$ ,  $((W)_\alpha)$ ,  $((\text{BIP})_\alpha)$  and  $((\text{BR})_\alpha)$ . Then for a general  $f \in \mathcal{H}_2^\alpha$ , one considers a dyadic decomposition  $f(\cdot) = \sum_{n \in \mathbb{Z}} f \cdot \phi(2^{-n}\cdot)$  for an appropriate  $\phi \in C^\infty(\frac{1}{2}, 2)$  and “puts the pieces  $f\phi(2^{-n}\cdot)$  together again” by techniques related to the Littlewood-Paley theory.

In this paper, we explore an operator theoretic approach to spectral multiplier theorems. In particular, we show that the “Paley-Littlewood arguments” for singular integrals can be replaced by a localization argument using the holomorphic  $H^\infty$ -calculus of the operator  $A$  (see Section 3). For many Laplace type operators, the boundedness of the  $H^\infty(\Sigma_\sigma)$  calculus is already well known. The various kernel bounds such as (generalized) Gaussian and Poisson bounds or maximal estimates which are commonly used in the first step, we replace by  $R$ -bounds: A set  $\tau$  of operators on an  $L^q(U)$ -space is called  $R$ -bounded if there is a constant  $C$  such that for all  $T_1, \dots, T_n \in L^q(U)$  and  $x_1, \dots, x_n \in L^q(U)$ , we have

$$\left\| \left( \sum_{j=1}^n |T_j x_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q(U)} \leq C \left\| \left( \sum_{j=1}^n |x_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q(U)}$$

(the smallest  $C$  in this inequality will be the  $R$ -bound  $R(\tau)$  of  $\tau$ ). Clearly such estimates are intimately related to Littlewood-Paley theory and it is known that many (generalized) Gaussian- and Poisson estimates, also on metric measure spaces with the doubling property (see e.g. [4, 5, 43]), as well as maximal estimates imply the above  $R$ -boundedness condition.

Therefore our operator theoretic approach allows us to present a unified approach to spectral multiplier theorems. Furthermore, we are not restricted to the  $L^q$ -scale and self-adjoint operators on  $L^2$ , but we can formulate theorems for 0-sectorial operators on a Banach space. Here are some samples. (From now on we assume  $\alpha \in (0, \infty)$ , not just  $\alpha \in \mathbb{N}$ .)

**Theorem 1.1.** (see Theorems 6.1, 7.1, 7.4). Let  $A$  be a 0-sectorial operator on a space  $L^q(U)$ ,  $1 < q < \infty$  (more generally on a Banach space  $X$  with Pisier's property  $(\alpha)$ ). Suppose furthermore that  $A$  has a bounded  $H^\infty(\Sigma_\omega)$  calculus for some  $\omega \in (0, \frac{\pi}{2})$ .

- (1) The  $R$ -boundedness of one of the sets  $(S)_\alpha, (W)_\alpha$  or  $(\text{BIP})_\alpha$  above implies that  $A$  has an  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus for  $\beta > \alpha + \frac{1}{2}$  on  $L^q(U)$ , i.e. the set

$$(1.4) \quad \left\{ f(A) : \|f\|_{\mathcal{H}_2^\beta} \leq 1 \right\} \text{ is } R\text{-bounded.}$$

- (2) Conversely, (1.4) implies that each of the sets  $(S)_\alpha, (W)_\alpha, (\text{BIP})_\alpha$  is  $R$ -bounded with  $\alpha \geq \beta$  ( $\alpha > \beta$  for the imaginary powers).

Here we used for  $\alpha > \frac{1}{p}$  the notation

$$\mathcal{H}_p^\alpha = \{f \in L_{loc}^p(\mathbb{R}_+) : \|f\|_{\mathcal{H}_p^\alpha} = \sup_{t>0} \|\phi f(t \cdot)\|_{W_p^\alpha(\mathbb{R})} < \infty\},$$

where  $\phi$  is a non-zero  $C_c^\infty(0, \infty)$  function (different choices resulting in equivalent norms) and  $W_p^\alpha(\mathbb{R})$  stands for the usual Sobolev space.

Some variants of this theorem concerning  $\mathcal{H}_p^\beta$  calculi with  $p \neq 2$  and “dyadic” bounds will be discussed in Sections 6 and 7. Essentially, these theorems say that in the presence of a bounded  $H^\infty$ -calculus,  $\mathcal{H}_2^\beta$  spectral multiplier theorems follow if one strengthens the norm bounds in  $(S)_\alpha, (W)_\alpha, (\text{BIP})_\alpha$  to  $R$ -bounds. We also show that the norm bounds for  $(S)_\alpha$  or  $(W)_\alpha$  by themselves are not strong enough to ensure spectral multiplier theorems (see Subsection 8.3). For  $(\text{BIP})_\alpha$  we have a positive result in Theorem 6.1. For functional calculi derived from the norm bounds in  $(S)_\alpha$ , see [27]. Due to the generality of our approach we do not always obtain the best possible exponent  $\beta$  for a given operator  $A$  with additional structure. However in our general setting our assumptions, in particular the gap between the parameter  $\alpha$  in  $(S)_\alpha, (W)_\alpha$  or  $(\text{BIP})_\alpha$  to the order  $\beta$  of the Hörmander calculus are close to being optimal; we discuss this in Section 8. Moreover, in the case of Bochner-Riesz means we obtain a characterization of the  $\mathcal{H}_1^\alpha$  calculus in terms of  $R$ -bounds as follows.

**Theorem 1.2.** Let  $A$  be as in Theorem 1.1 above.

- (1) If  $A$  has an  $R$ -bounded  $\mathcal{H}_1^\beta$  calculus, then

$$(1.5) \quad \{(1 - A/u)_+^{\alpha-1} : u > 0\} \text{ is } R\text{-bounded}$$

for all  $1 < \beta < \alpha$ .

- (2) Conversely, (1.5) implies that  $A$  has an  $R$ -bounded  $\mathcal{H}_1^\alpha$  functional calculus.

Our framework also allows us to work out a rather general version of the Paley-Littlewood theory for operators with a  $\mathcal{H}_p^\alpha$  calculus, including estimates of the form

(for  $X = L^q(U)$ ):

$$(1.6) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |\phi(2^n A)x|^2 \right)^{\frac{1}{2}} \right\| \cong \|x\|_{L^q},$$

$$(1.7) \quad \left\| \left( \int_0^\infty |\phi(tA)x|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\| \cong \|x\|_{L^q},$$

and their analogues in Banach spaces, see [38]. For precise characterizations of the exponents of the Hörmander calculus in terms of  $R$ -bounds and square functions of the form (1.6), (1.7), also in Banach spaces, see [39, 36].

We end this introduction with an overview of the article. In Section 2 we recall the definitions of  $R$ -boundedness and its variants and give some abstract results used in the main part of the paper. Section 3 contains the background on the holomorphic functional calculus as well as several function spaces related to (1.2). That is, we introduce the Sobolev and Hörmander function spaces  $\mathcal{W}_p^\alpha$  and  $\mathcal{H}_p^\alpha$  as well as the associated functional calculi, and show some simple properties used in this paper. In Section 4, we introduce and study an auxiliary Hörmander calculus. It allows to define  $f(A)$  for  $f$  locally belonging to  $\mathcal{H}_p^\alpha$  in a precised sense, under several possible weak assumptions. This is useful if there is no a priori self-adjoint calculus at hand, i.e. the underlying Banach space is not an  $L^p$  space, on which  $f(A)$  would be extended by density of  $L^2 \cap L^p$ . The issue of obtaining estimates of  $f(A)$  for  $f \in \mathcal{H}_p^\alpha$  from estimates for  $C_c^\infty(0, \infty)$  functions is addressed by means of a localization procedure of the support of  $f$ . Namely in Section 5, we show that under the presence of an  $H^\infty$  calculus, an  $R$ -bounded  $\mathcal{W}_p^\alpha$  calculus extends automatically to a  $\mathcal{H}_p^\alpha$  calculus (see Theorem 5.1). In Sections 6 and 7, we discuss the full  $\mathcal{H}_p^\alpha$  calculus and prove in particular the above Theorem 1.1. In Section 8, we discuss to what extent the assumptions of Theorem 1.1 are optimal and give several examples and counterexamples in connection with Theorem 1.1. In Section 9, we prove Theorem 1.2. Finally, in Section 10, we show how the results from Sections 6, 7 and 9 can be transferred to bisectorial operators. We also look at strip-type operators, which generate polynomially bounded groups and give a sketch of their theory of Hörmander type functional calculus using the results from the preceding sections.

## 2. $R$ -BOUNDED SETS OF OPERATORS

A classical theorem of Marcinkiewicz and Zygmund states that for elements  $x_1, \dots, x_n \in L^p(U, \mu)$  we can express “square sums” in terms of random sums

$$\left\| \left( \sum_{j=1}^n |x_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(U)} \cong \left( \mathbb{E} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|_{L^p(U)}^q \right)^{\frac{1}{q}} \cong \left( \mathbb{E} \left\| \sum_{j=1}^n \gamma_j x_j \right\|_{L^p(U)}^q \right)^{\frac{1}{q}}$$

with constants only depending on  $p, q \in [1, \infty)$ . Here  $(\epsilon_j)_j$  is a sequence of independent Bernoulli random variables (with  $P(\epsilon_j = 1) = P(\epsilon_j = -1) = \frac{1}{2}$ ) and  $(\gamma_j)_j$  is a sequence of independent standard Gaussian random variables. Following [8] it has become standard by now to replace square functions in the theory of Banach space valued function spaces by such random sums (see e.g. [43]). Note however that Bernoulli sums and Gaussian sums for  $x_1, \dots, x_n$  in a Banach space  $X$  are only equivalent if  $X$  has finite cotype (see [18, p. 218] for details).

**Definition 2.1.** Let  $\tau$  be a subset of  $B(X, Y)$ , where  $X$  and  $Y$  are Banach spaces. We say that  $\tau$  is  $R$ -bounded if there exists a  $C < \infty$  such that

$$\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k T_k x_k \right\| \leq C \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|$$

for any  $n \in \mathbb{N}$ ,  $T_1, \dots, T_n \in \tau$  and  $x_1, \dots, x_n \in X$ . The smallest admissible constant  $C$  is denoted by  $R(\tau)$ .

Recall that by definition,  $X$  has Pisier's property  $(\alpha)$  if for any finite family  $x_{k,l}$  in  $X$ ,  $(k, l) \in F$ , where  $F \subset \mathbb{Z} \times \mathbb{Z}$  is a finite array, we have a uniform equivalence

$$\mathbb{E}_\omega \mathbb{E}_{\omega'} \left\| \sum_{(k,l) \in F} \epsilon_k(\omega) \epsilon_l(\omega') x_{k,l} \right\|_X \cong \mathbb{E}_\omega \left\| \sum_{(k,l) \in F} \epsilon_{k,l}(\omega) x_{k,l} \right\|_X.$$

Note that property  $(\alpha)$  is inherited by closed subspaces, and that an  $L^p$  space has property  $(\alpha)$  provided  $1 \leq p < \infty$  [43, Section 4].

Recall that by definition,  $X$  has type  $p \in [1, 2]$  (resp. cotype  $q \in [2, \infty]$ ) if there is a uniform estimate for any finite family  $x_1, \dots, x_n$  in  $X$

$$\mathbb{E} \left\| \sum_k \epsilon_k x_k \right\|_X \lesssim \left( \sum_k \|x_k\|^p \right)^{\frac{1}{p}} \quad \text{resp.} \quad \left( \sum_k \|x_k\|^q \right)^{\frac{1}{q}} \lesssim \mathbb{E} \left\| \sum_k \epsilon_k x_k \right\|_X$$

(standard modification if  $q = \infty$ ). An  $L^p$  space for  $1 \leq p < \infty$  has always type  $\min(2, p)$  and cotype  $\max(2, p)$  [18, p. 219].

**Definition 2.2.** Let  $\tau \subset B(X, Y)$ . Then  $\tau$  is called semi- $R$ -bounded if there exists a  $C < \infty$  such that

$$\left( \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k T_k a_k x \right\|^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \|x\|$$

for any  $n \in \mathbb{N}$ ,  $T_1, \dots, T_n \in \tau$ ,  $a_1, \dots, a_n \in \mathbb{C}$  and  $x \in X$ . The least admissible constant  $C$  is denoted by  $R_s(\tau)$ . One clearly has that any  $R$ -bounded set is semi- $R$ -bounded and  $R_s(\tau) \leq R(\tau)$ .

We have the following characterization of semi- $R$ -boundedness in terms of  $R$ -boundedness.

**Lemma 2.3.** [61, Lemma 2.1] A set  $\tau \subset B(X, Y)$  is semi- $R$ -bounded with  $R_s(\tau) \leq M$  if and only if for all  $x \in X$ , the set  $\tau_x = \{Tx \in B(\mathbb{C}, X) : T \in \tau\}$  is  $R$ -bounded with  $R(\tau_x) \leq M\|x\|$ .

**Remark 2.4.** For a family  $\tau \subset B(X)$ , one always has  $R(\tau) \geq \sup_{T \in \tau} \|T\|$  and equivalence holds if and only if  $X$  is isomorphic to Hilbert space. On the other hand, if  $X$  or  $X'$  has type 2, then any bounded family  $\tau$  is automatically semi- $R$ -bounded, which motivates the above definition. Indeed, this follows from [61, Proposition 2.2] and a dualization argument for the case that  $X'$  has type 2. Note that if  $\tau \subset B(X, Y)$  is semi- $R$ -bounded, then  $\{T' : T \in \tau\}$  is semi- $R$ -bounded in  $B(Y', X')$ .

An important result is the following proposition due to Hytönen and Veraar which produces  $R$ -bounded sets by integrating against  $L^{r'}(\mathbb{R})$  functions.

**Proposition 2.5.** [30, Proposition 4.1, Remark 4.2]

- (1) Let  $X, Y$  be Banach spaces and  $(\Omega, \mu)$  a  $\sigma$ -finite measure space. Let  $r \in [1, \infty)$  satisfy  $\frac{1}{r} > \frac{1}{\text{type } Y} - \frac{1}{\text{cotype } X}$ . Further let  $T \in L^r(\Omega, B(X, Y))$  or assume merely that  $\omega \in \Omega \mapsto N(t) \in B(X, Y)$  is strongly measurable with  $\|T(\cdot)\|_{B(X, Y)}$  dominated by a function in  $L^r(\Omega)$ . Denote  $r'$  the conjugated exponent to  $r$ .

Then the set

$$\tau = \{T_f : \|f\|_{L^{r'}(\Omega)} \leq 1\}$$

is  $R$ -bounded, where  $T_fx = \int_{\Omega} f(\omega)T(\omega)x d\omega$  and  $R(\tau) \lesssim \|T\|_{L^r(\Omega, B(X, Y))}$ .

- (2) Let  $Y$  have type  $p$  and let  $r \in [1, \infty)$  satisfy  $\frac{1}{r} > \frac{1}{p} - \frac{1}{2}$ . If  $\omega \in \Omega \mapsto N(t) \in B(X, Y)$  is strongly measurable and there exists  $C < \infty$  such that  $(\int_{\Omega} \|N(t)x\|^r dt)^{\frac{1}{r}} \leq C\|x\|$ , then the set

$$\tau = \{T_f : \|f\|_{L^{r'}(\Omega)} \leq 1\}$$

is semi- $R$ -bounded and  $R_s(\tau) \lesssim C$ .

*Proof.* Part (1) is proved in [30, Proposition 4.1, Remark 4.2]. Then part (2) follows from part (1) and Lemma 2.3 in the following way.  $\mathbb{C}$  has cotype 2,  $Y$  has type  $p$ , and by assumption,  $\frac{1}{r} > \frac{1}{p} - \frac{1}{2}$ . Then part (1) yields that  $R(\{T_fx : \|f\|_{L^{r'}(\Omega)} \leq 1\}) \leq M\|x\|$  in  $B(\mathbb{C}, Y)$ . Thus by Lemma 2.3,  $R_s(\{T_f : \|f\|_{L^{r'}(\Omega)} \leq 1\}) \leq M$  in  $B(X, Y)$ .  $\square$

To obtain  $R$ -bounds, it sometimes suffices to have simple norm bounds for an analytic operator family, which autoimproves to an  $R$ -bounded version. This is precised in the following lemma.

**Lemma 2.6.** Assume that  $X$  has type  $p$  and cotype  $q$ . Let  $F : \mathbb{C}_+ \rightarrow B(X)$  be an analytic function such that  $\|F(z)\| \leq C \left(\frac{|z|}{\text{Re } z}\right)^{\alpha}$  for any  $\text{Re } z > 0$  and some  $\alpha \geq 0$ . Then for  $\delta > \frac{1}{p} - \frac{1}{q}$  there is a constant  $C < \infty$  such that we have

$$\left\{ \left( \frac{\text{Re } z}{|z|} \right)^{\alpha+\delta} F(z) : \text{Re } z = \epsilon \right\}$$

is  $R$ -bounded for any  $\epsilon > 0$ , with  $R$ -bound less than  $C$ .

*Proof.* For  $\lambda = 2\epsilon + is$  we have by the Cauchy integral formula

$$\frac{F(\lambda)}{\lambda^{\alpha+\delta}} = \frac{1}{2\pi i} \int_{\text{Re } z = \epsilon} \frac{1}{z - \lambda} \frac{F(z)}{z^{\alpha+\delta}} dz.$$

Choose  $\delta > \frac{1}{r} > \frac{1}{p} - \frac{1}{q}$ . Then

$$\left( \int_{\text{Re } z = \epsilon} \left| \frac{1}{z - \lambda} \right|^{r'} dz \right)^{\frac{1}{r'}} = \left( \int_{\mathbb{R}} \frac{1}{(|t - s|^2 + \epsilon^2)^{r'/2}} dt \right)^{\frac{1}{r'}} = \left( \int_{\mathbb{R}} \frac{1}{(1 + |t/\epsilon|^2)^{r'/2}} \frac{dt}{\epsilon} \right)^{\frac{1}{r'}} \frac{\epsilon^{\frac{1}{r'}}}{\epsilon}.$$

Furthermore,

$$\begin{aligned} \left( \int_{\text{Re } z = \epsilon} \left\| \frac{F(z)}{z^{\alpha+\delta}} \right\|^r dz \right)^{\frac{1}{r}} &\leq \sup_{\text{Re } z = \epsilon} \left\| \frac{F(z)}{z^{\alpha}} \right\| \left( \int_{\text{Re } z = \epsilon} \frac{1}{|z|^{r\delta}} dz \right)^{\frac{1}{r}} \leq C\epsilon^{-\alpha} \left( \int_{\mathbb{R}} \frac{1}{(\epsilon^2 + |t|^2)^{r\delta/2}} dt \right)^{\frac{1}{r}} \\ &= C\epsilon^{-\alpha} \epsilon^{-\delta} \epsilon^{\frac{1}{r}} \left( \int_{\mathbb{R}} \frac{1}{(1 + |t|^2)^{r\delta/2}} dt \right)^{\frac{1}{r}}. \end{aligned}$$

Hence,

$$\left( \int_{\operatorname{Re} z = \epsilon} \left| \frac{1}{z - \lambda} \right|^{r'} dz \right)^{\frac{1}{r'}} \left( \int_{\operatorname{Re} z = \epsilon} \left\| \frac{F(z)}{z^{\alpha+\delta}} \right\|^r dz \right)^{\frac{1}{r}} \leq C \frac{\epsilon^{\frac{1}{r'}}}{\epsilon} \epsilon^{-\alpha-\delta} \epsilon^{\frac{1}{r}} \cong C (\operatorname{Re} \lambda)^{-\alpha-\delta}.$$

By Proposition 2.5 and the fact that  $\frac{1}{r} > \frac{1}{p} - \frac{1}{q}$ , it follows that  $\{F(z) \left(\frac{\operatorname{Re} z}{|z|}\right)^{\alpha+\delta} : \operatorname{Re} z = \epsilon\}$  is  $R$ -bounded, with a uniform  $R$ -bound in  $\epsilon > 0$ .  $\square$

As a corollary, we record

**Corollary 2.7.** Let  $X$  be a Banach space with type  $p$ , cotype  $q$  and let  $\frac{1}{r} > \frac{1}{p} - \frac{1}{q}$ . Let  $-A$  generate an analytic semigroup  $(e^{-zA})_{\operatorname{Re} z > 0}$ .

- (1) If  $\|e^{-zA}\| \leq C \left(\frac{|z|}{\operatorname{Re} z}\right)^\alpha$  for  $\operatorname{Re} z > 0$ , then there is a constant  $C < \infty$  such that

$$R \left( \left\{ \left( \frac{\operatorname{Re} z}{|z|} \right)^\beta e^{-zA} : \operatorname{Re} z = \epsilon \right\} \right) \leq C \text{ for } \beta > \alpha + \frac{1}{r} \text{ and any } \epsilon > 0.$$

- (2) If  $\|z(z - A)^{-1}\| \leq C \frac{|z|^\alpha}{|\operatorname{Im} z|^\alpha}$  for  $\operatorname{Im} z > 0$ , then there is a  $C < \infty$  such that

$$R \left( \left\{ \left( \frac{|\operatorname{Im} z|}{|z|} \right)^\beta z(z - A)^{-1} : \operatorname{Im} z = \epsilon \right\} \right) \leq C \text{ for } \beta > \alpha + \frac{1}{r} \text{ and any } \epsilon \neq 0.$$

*Proof.* For (1), we set  $F(z) = \exp(-zA)$  and apply Lemma 2.6, whereas for (2), we set both  $F(z) = iz(iz - A)^{-1}$  and  $F(z) = -iz(-iz - A)^{-1}$  and apply Lemma 2.6 twice.  $\square$

In the claim of Lemma 2.6, one cannot replace the vertical axes  $\operatorname{Re} z = \epsilon$  by the right half plane  $\operatorname{Re} z > 0$ . This follows from the following counterexample.

**Example 2.8.** Let  $A$  be the negative generator of a bounded analytic semigroup which is not  $R$ -sectorial (see the beginning of Section 3 for the definition of  $R$ -sectoriality), i.e. for some  $\delta \in (0, \frac{\pi}{2})$ ,  $\{\exp(-zA) : z \in \Sigma_\delta\}$  is bounded, but for no  $\delta \in (0, \frac{\pi}{2})$  is  $\{\exp(-zA) : z \in \Sigma_\delta\}$   $R$ -bounded. For an example of such a semigroup on an  $L^p$ -space, see [23, Theorem 6.5] and [43, 2.20 Theorem]. Put  $a = \delta/\frac{\pi}{2} \in (0, 1)$  and set  $F(z) = \exp(-z^a A)$ . Then the assumptions of Lemma 2.6 hold with  $\alpha = 0$ . If  $\left\{ \left( \frac{\operatorname{Re} z}{|z|} \right)^\beta F(z) : \operatorname{Re} z > 0 \right\}$  were  $R$ -bounded for some  $\beta > 0$ , then  $\{F(z) : z \in \Sigma_{\frac{\pi}{4}}\}$  would be  $R$ -bounded, so  $\{\exp(-zA) : z \in \Sigma_{\delta/2}\}$  would be  $R$ -bounded. This is a contradiction.

The following result of van Gaans will be useful.

**Proposition 2.9.** [25, Theorem 3.1] Let  $X, Y$  be Banach spaces such that  $X$  has cotype  $q$  and  $Y$  has type  $p$ . Let  $\tau_1, \tau_2, \dots$  be  $R$ -bounded sets in  $B(X, Y)$  such that  $C = (\sum_{k=1}^\infty R(\tau_k)^r)^{\frac{1}{r}}$  is finite with  $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ . Then the union  $\tau = \bigcup_{k=1}^\infty \tau_k$  is  $R$ -bounded with  $R(\tau) \lesssim C$ .

As a corollary, we have the following further method to pass from bounded sets to (semi-)  $R$ -bounded ones.

**Corollary 2.10.** Let  $\mathbb{R} \ni t \mapsto U(t) \in B(X)$  a (not necessarily strongly continuous) one parameter group on a Banach space  $X$  with type  $p$  and cotype  $q$  and let  $X'$  have type  $p'$ . Assume that  $\{(1 + |t|)^{-\alpha} U(t) : t \in \mathbb{R}\}$  is bounded for some  $\alpha \geq 0$ .

- (1) Assume that  $\{U(t) : t \in [0, 1]\}$  is  $R$ -bounded. Then  $\{(1 + |t|)^{-\beta}U(t) : t \in \mathbb{R}\}$  is  $R$ -bounded for  $\beta > \alpha + \frac{1}{p} - \frac{1}{q}$ .
- (2) Assume that  $\{U(t) : t \in [0, 1]\}$  is semi- $R$ -bounded. Then  $\{(1 + |t|)^{-\beta}U(t) : t \in \mathbb{R}\}$  is semi- $R$ -bounded for  $\beta > \alpha + \min(\frac{1}{p}, \frac{1}{p'}) - \frac{1}{2}$ .

*Proof.* For part (1), we write

$$\{(1 + |t|)^{-\beta}U(t) : t \in \mathbb{R}\} \subseteq C \operatorname{conv} \left( \bigcup_{n \in \mathbb{Z}} \{U(t) : t \in [0, 1]\} \circ \{(1 + |n|)^{-(\beta-\alpha)}(1 + |n|)^{-\alpha}U(n)\} \right),$$

where  $\operatorname{conv}$  stands for the convex hull. By [43, 2.13 Theorem], taking the convex hull does not increase the  $R$ -bound. Note that the last set is  $R$ -bounded as a singleton with  $R$ -bound  $\lesssim (1 + |n|)^{-(\beta-\alpha)}$ , which is  $\ell^r(\mathbb{Z})$ -summable for  $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ . Since  $\{U(t) : t \in [0, 1]\}$  is  $R$ -bounded by assumption, Proposition 2.9 yields the claim.

For part (2), we argue similarly; note that the composition of a semi- $R$ -bounded set after a singleton is again semi- $R$ -bounded. We use Proposition 2.9 together with Lemma 2.3, a dualization argument if  $p' > p$  and the fact that  $U(t)'$  is again a one parameter group.  $\square$

### 3. THE $H^\infty$ AND HÖRMANDER CALCULUS

Our approach to the Hörmander calculus is based on the  $H^\infty$  calculus.

**3.1. 0-sectorial operators.** We briefly recall standard notions of sectorial operators and the  $H^\infty$  calculus. For  $\omega \in (0, \pi)$  we let  $\Sigma_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\}$  be the sector around the positive half-axis of aperture angle  $2\omega$ . We further define  $H^\infty(\Sigma_\omega)$  to be the space of bounded holomorphic functions on  $\Sigma_\omega$ . This space is a Banach algebra when equipped with the norm  $\|f\|_{\infty, \omega} = \sup_{\lambda \in \Sigma_\omega} |f(\lambda)|$ .

A closed operator  $A : D(A) \subset X \rightarrow X$  is called  $\omega$ -sectorial, if the spectrum  $\sigma(A)$  is contained in  $\overline{\Sigma_\omega}$ ,  $R(A)$  is dense in  $X$  and

$$(3.1) \quad \text{for all } \theta > \omega \text{ there is a } C_\theta > 0 \text{ such that } \|\lambda(\lambda - A)^{-1}\| \leq C_\theta \text{ for all } \lambda \in \overline{\Sigma_\theta}^c.$$

Note that  $\overline{R(A)} = X$  along with (3.1) implies that  $A$  is injective. In the literature, in the definition of sectoriality, the condition  $\overline{R(A)} = X$  is sometimes omitted. Note that if  $A$  satisfies the conditions defining  $\omega$ -sectoriality except  $\overline{R(A)} = X$  on  $X = L^p(\Omega)$ ,  $1 < p < \infty$  (or any reflexive space), then there is a canonical decomposition  $X = \overline{R(A)} \oplus N(A)$ ,  $x = x_1 \oplus x_2$ , and  $A = A_1 \oplus 0$ ,  $x \mapsto Ax_1 \oplus 0$ , such that  $A_1$  is  $\omega$ -sectorial on the space  $\overline{R(A)}$  with domain  $D(A_1) = \overline{R(A)} \cap D(A)$ .

For an  $\omega$ -sectorial operator  $A$  and a function  $f \in H^\infty(\Sigma_\theta)$  for some  $\theta \in (\omega, \pi)$  that satisfies moreover an estimate  $|f(\lambda)| \leq C|\lambda|^\epsilon/|1 + \lambda|^{2\epsilon}$ , one defines the operator

$$(3.2) \quad f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda,$$

where  $\Gamma$  is the boundary of a sector  $\Sigma_\sigma$  with  $\sigma \in (\omega, \theta)$ , oriented counterclockwise. By the estimate of  $f$ , the integral converges in norm and defines a bounded operator. If moreover there is an estimate  $\|f(A)\| \leq C\|f\|_{\infty, \theta}$  with  $C$  uniform over all such functions, then  $A$  is said to have a bounded  $H^\infty(\Sigma_\theta)$  calculus. In this case, there exists a bounded homomorphism  $H^\infty(\Sigma_\theta) \rightarrow B(X)$ ,  $f \mapsto f(A)$  extending the Cauchy integral formula (3.2).



We refer to [13] for details. We call  $A$  0-sectorial if  $A$  is  $\omega$ -sectorial for all  $\omega > 0$ . Further,  $A$  is called  $R$ -sectorial if  $\{\lambda(\lambda - A)^{-1} : \lambda \in \overline{\Sigma_\theta^c}\}$  is  $R$ -bounded for some  $\theta \in (0, \pi)$  [43, p. 76]. In this case,  $\omega_R(A)$  is defined to be the infimum over all such  $\theta$ . Note that if  $X$  has property  $(\alpha)$  (see Section 2 for the definition), then a sectorial operator with bounded  $H^\infty$  calculus is always  $R$ -sectorial [43, 12.8 Theorem]. For the definition of  $R$ -boundedness see Section 2.

To build stronger functional calculi we recall the following function spaces.

**Definition 3.1.**

- (1) Let  $p \in [1, \infty)$  and  $\alpha > \frac{1}{p}$ . We define

$$\mathcal{W}_p^\alpha = \{f : (0, \infty) \rightarrow \mathbb{C} : \|f\|_{\mathcal{W}_p^\alpha} = \|f_e\|_{W_p^\alpha} < \infty\}$$

and equip it with the norm  $\|f\|_{\mathcal{W}_p^\alpha}$ . Here we write from now on

$$f_e : J \rightarrow \mathbb{C}, z \mapsto f(e^z)$$

for a function  $f : I \rightarrow \mathbb{C}$  such that  $I \subset \mathbb{C} \setminus (-\infty, 0]$  and  $J = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi, e^z \in I\}$ . Moreover,  $W_p^\alpha = \{f \in L^p(\mathbb{R}) : \|f\|_{W_p^\alpha} = \|(\hat{f}(t)(1 + |t|)^\alpha)^\vee\|_p < \infty\}$ . The spaces  $W_p^\alpha$  and  $\mathcal{W}_p^\alpha$  are Banach algebras with respect to pointwise multiplication if  $\alpha > \frac{1}{p}$ .

- (2) Let  $\psi$  be a fixed function in  $C_c^\infty(\mathbb{R}_+) \setminus \{0\}$ . We define the Hörmander class

$$\mathcal{H}_p^\alpha = \{f \in L_{\text{loc}}^p(\mathbb{R}_+) : \|f\|_{\mathcal{H}_p^\alpha} = \sup_{t>0} \|\psi f(t \cdot)\|_{W_p^\alpha} < \infty\}.$$

This definition does not depend on the particular choice of  $\psi$ , two different choices giving equivalent norms, see e.g. [21, p. 445].

We have the following elementary properties of Hörmander spaces. Their proofs may be found in [35, Propositions 4.8 and 4.9, Remark 4.16].

**Lemma 3.2.** Let  $p, q \in [1, \infty)$ .

- (1) The spaces  $\mathcal{W}_p^\alpha$  and  $\mathcal{H}_p^\alpha$  are Banach algebras.
- (2) Let  $\alpha > \frac{1}{q} > \frac{1}{p}$ ,  $\alpha \geq \beta + \frac{1}{q} - \frac{1}{p}$  and  $\sigma \in (0, \pi)$ . Then

$$H^\infty(\Sigma_\sigma) \hookrightarrow \mathcal{H}_p^\alpha \hookrightarrow \mathcal{H}_q^\alpha \hookrightarrow \mathcal{H}_p^\beta.$$

In particular, the choice of  $p$  in  $\mathcal{H}_p^\alpha$  is only relevant when one is looking for the best exponent  $\alpha$ .

- (3) For any  $t > 0$ , we have  $\|f\|_{\mathcal{H}_p^\alpha} = \|f(t \cdot)\|_{\mathcal{H}_p^\alpha}$ .

**Remark 3.3.** The name ‘‘Hörmander class’’ is justified by the following fact. The classical Hörmander condition with a parameter  $\alpha_1 \in \mathbb{N}$  reads as follows [32, (7.9.8)]:

$$(3.3) \quad \sum_{k=0}^{\alpha_1} \sup_{R>0} \int_{R/2}^{2R} |R^k f^{(k)}(t)|^p dt / R < \infty.$$

By the following lemma which is proved in [35, Proposition 4.11], the norm  $\|\cdot\|_{\mathcal{H}_p^\alpha}$  expresses condition (3.4) and generalizes the classical Hörmander condition (3.3).

**Lemma 3.4.** Let  $f \in L_{\text{loc}}^1(\mathbb{R}_+)$ . Consider the conditions

- (1)  $f$  satisfies (3.3),

(2)  $f$  satisfies

$$(3.4) \quad \sup_{n \in \mathbb{Z}} \|\psi_n f_e\|_{W_p^\alpha} < \infty,$$

where  $(\psi_n)_{n \in \mathbb{Z}}$  is an equidistant partition of unity. By this we mean the following: Let  $\psi \in C_c^\infty(\mathbb{R})$ . Assume that  $\text{supp } \psi \subset [-1, 1]$  and  $\sum_{n=-\infty}^{\infty} \psi(t - n) = 1$  for all  $t \in \mathbb{R}$ . For  $n \in \mathbb{Z}$ , we put  $\psi_n = \psi(\cdot - n)$  and call  $(\psi_n)_{n \in \mathbb{Z}}$  an equidistant partition of unity. One easily checks that (3.4) does not depend on the particular choice of  $(\psi_n)_{n \in \mathbb{Z}}$ .

$$(3) \quad \|f\|_{\mathcal{H}_p^\alpha} < \infty.$$

Then (1)  $\Rightarrow$  (2) if  $\alpha_1 \geq \alpha$  and (2)  $\Rightarrow$  (1) if  $\alpha \geq \alpha_1$ . Further, (2)  $\Leftrightarrow$  (3).

**3.2. Functional calculus for 0-sectorial operators.** In order to reduce the Hörmander calculus to the  $H^\infty$  calculus we will use the following approximation of  $\mathcal{W}_p^\beta$  and  $\mathcal{H}_p^\beta$  functions holomorphic in a sector. The following lemma which is proved in [35, Lemma 4.15] will be useful.

**Lemma 3.5.** Let  $p \in [1, \infty)$  and  $\beta > \frac{1}{p}$ . Then  $\bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_p^\beta$  is dense in  $\mathcal{W}_p^\beta$ . More precisely, if  $f \in \mathcal{W}_p^\beta$ ,  $\psi \in C_c^\infty$  such that  $\psi(t) = 1$  for  $|t| \leq 1$  and  $\psi_n = \psi(2^{-n}(\cdot))$ , then

$$(f_e * \check{\psi}_n) \circ \log \in \bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_p^\beta \text{ and } (f_e * \check{\psi}_n) \circ \log \rightarrow f \text{ in } \mathcal{W}_p^\beta.$$

Thus if  $f$  happens to belong to several  $\mathcal{W}_p^\beta$  spaces as above with different indices, then it can be simultaneously approximated by a holomorphic sequence in any of these spaces.

Lemma 3.5 enables to base the  $\mathcal{W}_p^\beta$  calculus on the  $H^\infty$  calculus.

**Definition 3.6.** Let  $A$  be a 0-sectorial operator,  $p \in [1, \infty)$  and  $\beta > \frac{1}{p}$ . We say that  $A$  has a (bounded)  $\mathcal{W}_p^\beta$  calculus if there exists a constant  $C > 0$  such that

$$\|f(A)\| \leq C \|f\|_{\mathcal{W}_p^\beta} \quad (f \in \bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_p^\beta).$$

In this case, by the density of  $\bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_p^\beta$  in  $\mathcal{W}_p^\beta$ , the algebra homomorphism  $u : \bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_p^\beta \rightarrow B(X)$  given by  $u(f) = f(A)$  can be continuously extended in a unique way to a bounded algebra homomorphism

$$u : \mathcal{W}_p^\beta \rightarrow B(X), f \mapsto u(f).$$

We write again  $f(A) = u(f)$  for any  $f \in \mathcal{W}_p^\beta$ . Assume that  $A$  has a  $\mathcal{W}_p^\beta$  calculus and a  $\mathcal{W}_{p'}^{\beta'}$  calculus. Then for  $f \in \mathcal{W}_p^\beta \cap \mathcal{W}_{p'}^{\beta'}$ ,  $f(A)$  is defined twice by the above. However, Lemma 3.5 shows that these definitions coincide.

**Definition 3.7.** Let  $A$  be a 0-sectorial operator. We say that  $A$  has an  $R$ -bounded  $\mathcal{W}_p^\alpha$  calculus if  $A$  has a  $\mathcal{W}_p^\alpha$  calculus, which is an  $R$ -bounded mapping in the sense of [40, Definition 2.7], i.e.

$$R(\{f(A) : \|f\|_{\mathcal{W}_p^\alpha} \leq 1\}) < \infty.$$

**Definition 3.8.** Let  $p \in [1, \infty)$ ,  $\alpha > \frac{1}{p}$  and let  $A$  be a 0-sectorial operator. We say that  $A$  has a (bounded)  $\mathcal{H}_p^\alpha$  calculus if there exists a constant  $C > 0$  such that

$$(3.5) \quad \|f(A)\| \leq C \|f\|_{\mathcal{H}_p^\alpha} \quad (f \in \bigcap_{\omega \in (0, \pi)} H^\infty(\Sigma_\omega) \cap \mathcal{H}_p^\alpha).$$

Similarly as in Definition 3.7, we say that  $A$  has an  $R$ -bounded  $\mathcal{H}_p^\alpha$  calculus if moreover  $R(\{f(A) : f \in \bigcap_{\omega \in (0, \pi)} H^\infty(\Sigma_\omega) \cap \mathcal{H}_p^\alpha, \|f\|_{\mathcal{H}_p^\alpha} \leq 1\}) < \infty$ .

Finally, we record some norm estimates which will be useful later. The functions in the following norm estimates correspond via functional calculus to typical operator families. We use the short hand notation  $\langle t \rangle = \sqrt{1 + t^2}$ .

**Lemma 3.9.** Let  $p \in [1, \infty)$  and  $\alpha > \frac{1}{p}$ . We have the following  $\mathcal{H}_p^\alpha$  norm estimates for functions depending on the variable  $\lambda > 0$ .

- (1) For  $\theta \in (-\pi, \pi)$ ,  $\|\exp(-e^{i\theta}\lambda)\|_{\mathcal{H}_p^\alpha} \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha}$ .
- (2) For  $s \in \mathbb{R}$ ,  $\|(1 + |s|\lambda)^{-\alpha} \exp(is\lambda)\|_{\mathcal{H}_p^\alpha} \lesssim 1$ .
- (3) For  $t \in \mathbb{R}$ ,  $\|\lambda^{it}\|_{\mathcal{H}_p^{\alpha-\epsilon}} \lesssim \langle t \rangle^\alpha$ .
- (4) For  $1 < \beta < \alpha$ ,  $\sup_{u>0} \|(1 - \lambda/u)_+^{\alpha-1}\|_{\mathcal{H}_1^\beta} < \infty$ , where  $x_+ = \max(x, 0)$ .

*Proof.* (1) Let first  $\alpha = n \in \mathbb{N}_0$  and put  $f(\lambda) = \exp(-e^{i\theta}\lambda)$ . We have by Lemma 3.4,  $\|f\|_{\mathcal{H}_p^n} \lesssim \sup\{|\lambda^k f^{(k)}(\lambda)| : k = 0, \dots, n, \lambda > 0\}$ . The latter can easily be estimated by  $(\frac{\pi}{2} - |\theta|)^{-n}$ . For non-integer  $\alpha = n + \vartheta$ ,  $\vartheta \in (0, 1)$ , we use the complex interpolation  $[W_p^n, W_p^{n+1}]_\vartheta = W_p^\alpha$  to deduce by Lemma 3.4

$$\begin{aligned} \|f\|_{\mathcal{H}_p^\alpha} &\cong \sup_{t>0} \|f(t \cdot) \psi\|_{W_p^\alpha} \leq \sup_{t>0} \|f(t \cdot) \psi\|_{W_p^n}^{1-\vartheta} \cdot \|f(t \cdot) \psi\|_{W_p^{n+1}}^\vartheta \\ &\lesssim (\frac{\pi}{2} - |\theta|)^{-[n(1-\vartheta) + (n+1)\vartheta]} = (\frac{\pi}{2} - |\theta|)^{-\alpha}. \end{aligned}$$

(2) For  $\operatorname{Re} z > 0$ , put  $g(z; \lambda) = (1 + |s|\lambda)^{-z} \exp(is\lambda)$ . Similarly to (1), it is easy to check that  $\|g(n + i\tau; \cdot)\|_{\mathcal{H}_p^n} \lesssim \langle \tau \rangle^n$  and  $\sup_{\lambda>0; k=0,1,\dots,n} \lambda^k \frac{d^k}{d\lambda^k} |g(n + i\tau; \lambda)| \lesssim \langle \tau \rangle^n$  for  $\tau \in \mathbb{R}$ . Then we can deduce from Stein's subexponential complex interpolation [55, Theorem 1] that  $\|g(\alpha; \cdot)\|_{\mathcal{H}_p^\alpha} \lesssim 1$  for general  $\alpha$ .

(3) For integer  $\alpha \geq 1$  and any  $p \in (1, \infty)$ , it is easy to check that  $\|\lambda \mapsto \lambda^{it}\|_{\mathcal{H}_p^\alpha} \lesssim \langle t \rangle^\alpha$ . Likewise, since  $\|\lambda \mapsto \lambda^{it}\|_{L^\infty(\mathbb{R}_+)} \lesssim 1$ , we have  $\|\lambda \mapsto \lambda^{it}\|_{\mathcal{H}_q^{\frac{1}{q}+\epsilon}} \lesssim 1$  for any  $q \in (1, \infty)$  and  $\epsilon > 0$ . Now for  $\alpha > 1$ , apply complex interpolation similar to parts (1) and (2), between the cases  $\alpha_1 = \lfloor \alpha \rfloor$  and  $\alpha_2 = \alpha_1 + 1$ , and for  $\alpha < 1$ , between the  $\mathcal{H}_q^{\frac{1}{q}+\epsilon}$  estimate with  $q$  close to  $\infty$  and the  $\mathcal{H}_p^\alpha$  estimate with  $\alpha = 1$  and  $p = q$ .

(4) By Lemma 3.2 (3),  $\|(1 - \lambda/u)_+^{\alpha-1}\|_{\mathcal{H}_1^\beta} \cong \|(1 - \lambda)_+^{\alpha-1}\|_{\mathcal{H}_1^\beta}$ , so that it only remains to prove that  $(1 - \lambda)_+^{\alpha-1}$  belongs to  $\mathcal{H}_1^\beta$ . It remains to estimate  $\sup_{n \in \mathbb{Z}} \|(1 - 2^n \lambda)_+^{\alpha-1} \varphi_0(\lambda)\|_{W_1^\beta}$ , where  $\varphi_0 \in C_c^\infty$  such that  $\operatorname{supp} \varphi_0 \subset [\frac{1}{2}, 2]$ . Let first  $n \leq -2$  and  $m$  the least integer greater or equal than  $\beta$ . We have

$$\begin{aligned} \|(1 - 2^n \lambda)_+^{\alpha-1} \varphi_0(\lambda)\|_{W_1^\beta} &\leq \|(1 - 2^n \lambda)_+^{\alpha-1} \varphi_0(\lambda)\|_{W_1^m} \lesssim \max_{k=0,1,\dots,m} \int_{\frac{1}{2}}^2 \left| \frac{d^k}{d\lambda^k} (1 - 2^n \lambda)^{\alpha-1} \right| d\lambda \\ &\lesssim \max_{k=0,1,\dots,m} \int_{\frac{1}{2}}^2 2^{nk} (1 - 2^n \lambda)^{\alpha-1-k} d\lambda. \end{aligned}$$

Since  $n \leq -2$ , we have  $\frac{1}{2} \leq (1 - 2^n \lambda) \leq 1$  for  $\frac{1}{2} \leq \lambda \leq 2$ , and  $2^{nk} \leq 1$ , so that the above expression is uniformly bounded in  $n \leq -2$ . For  $n \geq 1$ , we have  $(1 - 2^n \lambda)_+^{\alpha-1} = 0$  for  $\lambda \geq \frac{1}{2} \geq 2^{-n}$ , so that  $(1 - 2^n \lambda)_+^{\alpha-1} \varphi_0(\lambda) \equiv 0$ . Finally for  $n \in \{-1, 0\}$ ,  $\|(1 - 2^n \lambda)_+^{\alpha-1} \varphi_0(\lambda)\|_{W_1^\beta} = \|(1 - 2^{2n} \lambda^2)_+^{\alpha-1} [(1 + 2^n \lambda)_+^{1-\alpha} \varphi_0(\lambda)]\|_{W_1^\beta}$ . According to [10, p. 11], one has  $(1 - \lambda^2)_+^{\alpha-1} \in W_1^\beta \iff \beta < \alpha$ , and the space  $W_1^\beta$  is invariant under dilations  $f \mapsto f(t \cdot)$ , so that the first factor  $(1 - 2^{2n} \lambda^2)_+^{\alpha-1}$  in the last expression belongs to  $W_1^\beta$ . Furthermore, the second expression in between the brackets belongs to  $C_c^\infty$ , so that the whole term belongs to  $W_1^\beta$ . The lemma is proven.  $\square$

#### 4. EXTENDED HÖRMANDER CALCULUS

In statements like  $(W)_\alpha$ ,  $(\text{BIP})_\alpha$  and  $(\text{BR})_\alpha$  in the introduction, one would like to give a meaning to “operators”  $f(A)$  such as  $\exp(isA)$ ,  $A^{it}$  and  $(1 - A/u)_+^\alpha$  before we have established the boundedness of a  $\mathcal{H}_p^\alpha$  calculus. In the classical case of a selfadjoint operator on  $L^2(U)$  and  $X = L^q(U)$ , one considers  $f(A)$  as defined by functional calculus on  $L^2$  and thinks of  $f(A)$  on  $L^q \cap L^2$  as the part of  $f(A)$  on  $L^q \cap L^2$ . In the formulas it is then implicitly assumed that the unbounded operator “ $f(A)$ ” has a continuous extension to an operator in  $B(L^q(U))$ . This assumption is then part of  $(W)_\alpha$ ,  $(\text{BIP})_\alpha$  and  $(\text{BR})_\alpha$ . In the general case of 0-sectorial operators on a Banach space one has to say unfortunately a little bit more to circumvent this purely formal difficulty. It is convenient to consider the subspaces  $D(\theta) = D(A^\theta) \cap R(A^\theta)$  for  $\theta > 0$ , which are Banach spaces with norm  $\|x\|_{D(\theta)} = \|\rho^{-\theta}(A)x\|_X$  and the  $D(\theta)$  form a decreasing sequence of spaces when  $\theta$  grows. Here  $\rho(\lambda) = \lambda(1+\lambda)^{-2}$  belongs together with its powers  $\rho^\theta$  to  $H_0^\infty(\Sigma_\omega)$  for any  $\omega \in (0, \pi)$ , and  $R(\rho^\theta(A)) = D(A^\theta) \cap R(A^\theta)$ . Note that  $D(\theta)$  is dense in  $X$  (see [43, 9.4 Proposition (c)] for the case  $\theta \in \mathbb{N}$ ). Then to make sense of  $f(A)$  we will use an “extended” version of the  $H^\infty$  and  $\mathcal{H}_p^\alpha$  calculus which only produces closable operators on  $D(\theta)$ .

For  $\omega \in (0, \pi)$ , define the algebras of functions  $\text{Hol}(\Sigma_\omega) = \{f : \Sigma_\omega \rightarrow \mathbb{C} : \exists n \in \mathbb{N} : \rho^n f \in H^\infty(\Sigma_\omega)\}$ . For a proof of the following lemma, we refer to [43, Section 15B] and [28, p. 91-96].

**Lemma 4.1.** Let  $A$  be a 0-sectorial operator. There exists a linear mapping, called the extended holomorphic calculus,

$$(4.1) \quad \bigcup_{\omega > 0} \text{Hol}(\Sigma_\omega) \rightarrow \{\text{closed and densely defined operators on } X\}, \quad f \mapsto f(A)$$

extending (3.2) such that for any  $f, g \in \text{Hol}(\Sigma_\omega)$ ,  $f(A)g(A)x = (fg)(A)x$  for  $x \in \{y \in D(g(A)) : g(A)y \in D(f(A))\} \subset D((fg)(A))$  and  $D(f(A)) = \{x \in X : (\rho^n f)(A)x \in D(A^n) \cap R(A^n) = D(n)\}$ , where  $(\rho^n f)(A)$  is given by (3.2), i.e.  $n \in \mathbb{N}$  is sufficiently large.

In an analogous way we introduce an extended  $\mathcal{H}_p^\alpha$  calculus  $\Phi_A : \mathcal{H}_p^\alpha \rightarrow B(D(\theta), X)$  for some  $\alpha > \frac{1}{p}$ ,  $\theta > 0$ . The existence of such a calculus is known in many concrete situations.

For Lemma 4.3 and the sequel, we need the following notion.

**Definition 4.2.** Let  $\varphi \in C_c^\infty(\mathbb{R}_+)$ . Assume that  $\text{supp } \varphi \subset [\frac{1}{2}, 2]$  and  $\sum_{n=-\infty}^\infty \varphi(2^{-n}t) = 1$  for all  $t > 0$ . For  $n \in \mathbb{Z}$ , we put  $\varphi_n = \varphi(2^{-n} \cdot)$  and call  $(\varphi_n)_{n \in \mathbb{Z}}$  a dyadic partition

of unity. For the existence of such a partition, we refer to the idea in [2, Lemma 6.1.7].

**Lemma 4.3.** Let  $A$  be a 0-sectorial operator on a Banach space  $X$ . Assume that one of the following conditions holds.

- (1)  $\|A^{it}\|$  is polynomially bounded in  $t \in \mathbb{R}$ .
- (2)  $\|\exp(-zA)\| \leq C \left(\frac{|z|}{\operatorname{Re} z}\right)^\beta$  for some  $\beta > 0$  and all  $z \in \mathbb{C}_+$ .
- (3)  $\|\lambda R(\lambda, A)\| \leq C |\arg(\lambda)|^{-\beta}$  for some  $\beta > 0$  and all  $\lambda \in \mathbb{C} \setminus [0, \infty)$ .

Then for  $p \in [1, 2]$  and some suitable  $\alpha > \frac{1}{p}$  and  $\theta > 0$ , there exists an auxiliary functional calculus  $\Phi_A : \mathcal{H}_p^\alpha \rightarrow B(D(\theta), X)$ , which is a linear mapping and has the compatibility that for  $\omega \in (0, \pi)$ ,  $f \in H^\infty(\Sigma_\omega) \cap \mathcal{H}_p^\alpha$  and  $x \in D(\theta)$ , there holds  $\Phi_A(f)x = f(A)x$ , where the right hand side is defined by the holomorphic functional calculus from Lemma 4.1. Moreover,  $\Phi_A(f)$  is a closable operator over  $X$ . We can denote without ambiguity  $f(A)$  its closure. Then we have the further compatibilities

- (1) If  $f \in \mathcal{H}_p^\alpha$ ,  $g \in H^\infty(\Sigma_\omega)$  and  $x \in D(\theta)$ , then  $f(A)g(A)x = (fg)(A)x$ .
- (2) If  $f \in \mathcal{H}_p^\alpha$ ,  $g \in H_0^\infty(\Sigma_\omega)$  and  $x \in D(\theta)$ , then  $g(A)f(A)x = (gf)(A)x$ .
- (3) If  $f, g \in \mathcal{H}_p^\alpha$  and  $x \in D(\theta)$ , then  $g(A)x \in D(f(A))$  and  $f(A)g(A)x = (fg)(A)x$ .

*Proof.* The Lemma is proved for the case  $p = 2$  and a mapping  $\Phi_A^{\mathcal{W}_p^\alpha} : \mathcal{W}_p^\alpha \rightarrow B(D(\theta), X)$ , for the imaginary powers in [39], for the resolvents in [38, Proposition 3.9], and for the semigroup, see this paper at Remark 7.2. The case of general  $p$  is entirely similar. To pass from  $\mathcal{W}_p^\alpha$  functions to  $\mathcal{H}_p^\alpha$  functions, note that for  $f \in \mathcal{H}_p^\alpha$  and  $\nu > 0$ , we have  $f\rho^\nu \in \mathcal{W}_p^\alpha$ . Indeed, with  $(\varphi_k)_{k \in \mathbb{Z}}$  a dyadic partition of unity and  $\widetilde{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ , we have

$$\begin{aligned} \|f\rho^\nu\|_{\mathcal{W}_p^\alpha} &\leq \sum_{k \in \mathbb{Z}} \|f\rho^\nu \varphi_k\|_{\mathcal{W}_p^\alpha} \\ &\lesssim \sum_{k \in \mathbb{Z}} \|f\varphi_k\|_{\mathcal{W}_p^\alpha} \|\rho^\nu \widetilde{\varphi}_k\|_{\mathcal{W}_p^\alpha} \\ &\lesssim \|f\|_{\mathcal{H}_p^\alpha} \sum_{k \in \mathbb{Z}} 2^{-|k|\nu} \lesssim \|f\|_{\mathcal{H}_p^\alpha}. \end{aligned}$$

Now suppose that  $\Phi_A^{\mathcal{W}_p^\alpha} : \mathcal{W}_p^\alpha \rightarrow B(D(\theta), X)$  is an auxiliary calculus as above. Let  $\theta' > \theta$  and set  $\nu = \theta' - \theta > 0$ . Then  $\Phi_A : \mathcal{H}_p^\alpha \rightarrow B(D(\theta'), X)$ ,  $f \mapsto (x = \rho^\nu(A)y \mapsto \Phi_A^{\mathcal{W}_p^\alpha}(f\rho^\nu)y)$  is the desired auxiliary calculus of the lemma, where  $y \in D(\theta)$ . Now it is easy to check that the already established compatibilities of  $\Phi_A^{\mathcal{W}_p^\alpha}$  carry over to  $\Phi_A$ .  $\square$

In some cases, it is convenient to consider an even smaller domain of definition than  $D(\theta)$  : For a 0-sectorial operator  $A$  with auxiliary functional calculus  $\Phi_A$  as above, we define the following subset  $D_A$  of  $X$ . Let  $(\varphi_n)_{n \in \mathbb{Z}}$  be a dyadic partition of unity.

$$(4.2) \quad D_A = \left\{ \sum_{n=-N}^N \varphi_n(A)x : N \in \mathbb{N}, x \in D(\theta) \right\}.$$

We call  $D_A$  the calculus core of  $A$ . According to [39],  $D_A$  is dense in  $X$ .

As for the  $H^\infty$  calculus, there is an extended  $\mathcal{H}_p^\alpha$  calculus which is defined for  $f : (0, \infty) \rightarrow \mathbb{C}$  with  $f\rho^\nu = f(\cdot)(\cdot)^\nu(1+(\cdot))^{-2\nu} \in \mathcal{H}_p^\alpha$  for some  $\nu > 0$ , as a counterpart of (4.1).

**Definition 4.4.** Let  $A$  have an auxiliary calculus  $\Phi_A : \mathcal{H}_p^\alpha \rightarrow B(D(\theta), X)$ . Let  $f : (0, \infty) \rightarrow \mathbb{C}$  with  $f\rho^\nu \in \mathcal{H}_p^\alpha$  for some  $\nu > 0$ . We define the operator  $f(A)$  on  $D_A$  by

$$f(A)\left(\sum_{n=-N}^N \varphi_n(A)x\right) = \sum_{n=-N}^N (f\varphi_n)(A)x.$$

Note that this definition does not depend on the representation  $\sum_{n=-N}^N \varphi_n(A)x$  of the element in  $D_A$ .

**Lemma 4.5.** Let  $1 \leq p < \infty$  and  $A$  have an auxiliary calculus  $\Phi_A : \mathcal{H}_p^\alpha \rightarrow B(D(\theta), X)$  such that  $\Phi_A(f)x = f(A)x$  for  $x \in D(\theta)$  and  $f \in H^\infty(\Sigma_\omega) \cap \mathcal{H}_p^\alpha$ . Assume that  $f\rho^\nu \in \mathcal{H}_p^\alpha$  for some  $\nu > 0$ , and  $g$  a further function, where we suppose the same assumptions as  $f$ .

- (a) The operator  $f(A)$  is closable. We denote the closure by slight abuse of notation again by  $f(A)$ .
- (b) If furthermore  $f \in \mathcal{H}_p^\alpha$  then  $f(A)$  coincides with the operator defined by the calculus  $\Phi_A$ . If  $f \in \text{Hol}(\Sigma_\omega)$  for some  $\omega \in (0, \pi)$ , then  $f(A)$  coincides with the (unbounded) holomorphic calculus of  $A$ .
- (c) For any  $x \in D_A$ , we have  $g(A)x \in D(f(A))$  and  $f(A)g(A)x = (fg)(A)x$ .

*Proof.* This is proved in [39] in the case  $p = 2$  and for  $\mathcal{W}_p^\alpha$  in place of  $\mathcal{H}_p^\alpha$ . Then the case  $p \in [1, 2]$  is entirely similar, and for the  $\mathcal{H}_p^\alpha$  case, we can proceed as in the proof of Lemma 4.3 noting that  $f\rho^\nu \in \mathcal{H}_p^\alpha$  implies  $f\rho^{\nu'} \in \mathcal{W}_p^\alpha$  for  $\nu' > \nu > 0$ .  $\square$

Assume that  $A$  has a  $\mathcal{H}_p^\alpha$  calculus in the sense of Definition 3.8. Then it has an auxiliary functional calculus as in Lemma 4.5 and the estimate (3.5) extends to all of  $f \in \mathcal{H}_p^\alpha$ .

The following lemma gives several representation formulas for the  $\mathcal{W}_p^\alpha$  calculus and the auxiliary  $\mathcal{H}_p^\alpha$  calculus, in terms of the  $C_0$ -group  $A^{it}$ , the wave group  $e^{itA}$  of (in general) unbounded operators and the Bochner-Riesz means  $(1 - A/u)_+^{\alpha-1}$ .

**Lemma 4.6.** Let  $X$  be a Banach space with dual  $X'$ , and let  $p \in [1, 2]$ . Let  $\alpha > \frac{1}{p}$ , so that  $\mathcal{W}_p^\alpha$  is a Banach algebra. Let  $A$  be a 0-sectorial operator with imaginary powers  $A^{it}$ .

- (1) Assume that for some  $C > 0$  and all  $x \in X$ ,  $x' \in X'$

$$(4.3) \quad \left( \int_{\mathbb{R}} |t|^{-\alpha} |\langle A^{it}x, x' \rangle|^p dt \right)^{1/p} \leq C \|x\| \|x'\|.$$

Then  $A$  has a bounded  $\mathcal{W}_p^\alpha$  calculus. Moreover, for any  $f \in \mathcal{W}_p^\alpha$ ,  $f(A)$  is given by

$$\langle f(A)x, x' \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{(f_e)}(t) \langle A^{it}x, x' \rangle dt \quad (x \in X, x' \in X').$$

The above integral exists as a strong integral if moreover  $\langle t \rangle^{-\alpha} A^{it}x \in L^p(\mathbb{R}; X)$ .

- (2) Conversely, if  $p = 2$  and  $A$  has a  $\mathcal{W}_2^\alpha$  calculus, then (4.3) holds.

- (3) Assume that  $A$  has an auxiliary calculus  $\Phi_A : \mathcal{H}_p^\gamma \rightarrow B(D(\theta), X)$  for some  $\theta \geq 0$  and  $\gamma > \frac{1}{p} \geq \frac{1}{2}$ . Then for  $x$  belonging to the calculus core  $D_A$  of  $A$  and for  $f \in C_c^\infty(0, \infty)$ ,

$$f(A)x = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itA} x dt.$$

- (4) Assume that  $1 < \beta < \alpha$  and that  $A$  has an auxiliary calculus  $\Phi_A : \mathcal{H}_1^\beta \rightarrow B(D(\theta), X)$  for some  $\theta \geq 0$  and an  $H^\infty(\Sigma_\omega)$  calculus for some  $\omega \in (0, \frac{\pi}{2})$ . Then the Bochner-Riesz means  $R_u^{\alpha-1}(A)$  are densely defined closed operators, where  $R_u^{\alpha-1}(\lambda) = (1 - \lambda/u)_+^{\alpha-1} \in \mathcal{H}_1^\beta$  and  $t_+ = \max(t, 0)$ . Moreover, for  $f \in C_c^\infty(\mathbb{R}_+)$  with  $\text{supp } f \subset [\frac{1}{2}, 2]$  and  $x \in D_A$ , one has

$$f(A)x = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty f^{(\alpha)}(u) u^{\alpha-1} R_u^{\alpha-1}(A) x du,$$

where  $f^{(\alpha)}$  is defined e.g. by  $f^{(\alpha)}(\xi) = (-i\xi)^\alpha \hat{f}(\xi)$ ,  $\xi \in \mathbb{R}$ , and  $m = \lfloor \alpha \rfloor$ .

The same formula holds if  $X = L^p(U, \mu)$  for some  $1 < p < \infty$ ,  $A$  is self-adjoint on  $L^2(U, \mu)$ , and  $R_u^{\alpha-1}(A)$  is densely defined on  $L^p$  by the self-adjoint spectral calculus for  $x \in L^2 \cap L^p$ .

*Proof.* (1) and (2) This can be proved with the Cauchy integral formula (3.2) in combination with the Fourier inversion formula, see [35, Proposition 4.22].

(3) Let  $\phi \in C_c^\infty(0, \infty)$ . By the Fourier inversion formula, one has

$$(4.4) \quad f(\lambda)\phi(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{it\lambda} \phi(\lambda) dt,$$

which is in particular a Bochner integral formula with values in the space  $\mathcal{H}_p^\gamma$ , since  $\|\lambda \mapsto e^{it\lambda} \phi(\lambda)\|_{\mathcal{W}_p^\gamma} \lesssim \langle t \rangle^\gamma$  and  $\hat{f}(t)$  is rapidly decreasing. If  $x = \sum_{n=-N}^N \varphi_n(A) y \in D_A$ , then let  $\phi = \sum_{n=-N}^N \varphi_n$ . Applying the auxiliary  $\mathcal{H}_p^\gamma$  calculus on both sides of (4.4) yields (3).

(4) For  $f$  as above and  $\phi \in C_c^\infty(0, \infty)$ , one has the representation formula (4.5)

$$f(s)\phi(s) = \frac{(-1)^m}{\Gamma(\alpha)} \int_s^\infty (t-s)^{\alpha-1} f^{(\alpha)}(t) \phi(s) dt = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty (t-s)_+^{\alpha-1} f^{(\alpha)}(t) \phi(s) dt, \quad s > 0,$$

where  $m = \lfloor \alpha \rfloor$  [26, p. 1011]. This implies that if  $f(s) = 0$  for  $s \geq r$ , then  $f^{(\alpha)}(s) = 0$  for  $s \geq r$ . It is now easy to check that the integral formula (4.5) holds as a Bochner integral in  $\mathcal{H}_1^\beta$ , and applying the auxiliary  $\mathcal{H}_1^\beta$  calculus yields with  $\phi \in C_c^\infty(0, \infty)$  such that  $x = \phi(A)y$ ,  $y \in D(\theta)$ ,

$$f(A)x = (f\phi)(A)y = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty f^{(\alpha)}(t) (t-A)_+^{\alpha-1} \phi(A)y dt = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty f^{(\alpha)}(u) u^{\alpha-1} R_u^{\alpha-1}(A) x dt.$$

The case  $X = L^p(U, \mu)$ ,  $A$  self-adjoint on  $L^2$  and  $x \in L^p(U) \cap L^2(U)$  also follows from (4.5), regarded as a Bochner integral in  $\mathcal{B}^\infty(\mathbb{R}_+)$  by applying the self-adjoint calculus.  $\square$

## 5. THE LOCALIZATION PRINCIPLE AND $\mathcal{H}_p^\alpha$ CALCULUS

In this section we reduce the problem of showing the boundedness of the  $\mathcal{H}_p^\alpha$  calculus to boundedness on functions with compact support, by a localization principle. We are going to show that in the presence of some  $H^\infty$  calculus, an  $R$ -bounded  $\mathcal{W}_p^\alpha$  calculus can be improved to an ( $R$ -bounded)  $\mathcal{H}_p^\alpha$  calculus. This replaces the Littlewood-Paley arguments in the proof of classical spectral multiplier theorems. Indeed, a somewhat weaker assumption than  $R$ -bounded  $\mathcal{W}_p^\alpha$  calculus suffices, as is shown in the following theorem.

**Theorem 5.1.** Let  $A$  be a 0-sectorial operator having an  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \pi)$  and assume that  $X$  has property  $(\alpha)$ . Consider the following conditions for  $p \in [1, \infty)$  and  $\alpha > \frac{1}{p}$ .

- (1)  $A$  has an  $R$ -bounded  $\mathcal{W}_p^\alpha$  calculus.
  - (2)  $A$  has an auxiliary calculus  $\Phi_A : \mathcal{H}_p^\beta \rightarrow B(D(\theta), X)$  for some  $\beta, \theta > 0$ , (so that  $f(2^n A)$  is well-defined for  $f \in C_c^\infty(\mathbb{R}_+)$  and  $n \in \mathbb{Z}$ ), and we have
- (5.1)  $\left\{ f(2^n A) : f \in C_c^\infty(\mathbb{R}_+), \text{supp } f \subset [\frac{1}{2}, 2], \|f\|_{\mathcal{W}_p^\alpha} \leq 1, n \in \mathbb{Z} \right\}$  is  $R$ -bounded.
- (3)  $A$  has an  $R$ -bounded  $\mathcal{H}_p^\alpha$  calculus.

Then all these conditions are equivalent. If  $X$  does not have property  $(\alpha)$ , but  $A$  is  $R$ -sectorial, then one still has  $(1) \implies (2) \implies (3')$ , with  $(3')$ :  $A$  has a bounded  $\mathcal{H}_p^\alpha$  calculus.

*Proof.* Since  $\mathcal{W}_p^\alpha \subset \mathcal{H}_p^\alpha$ , we clearly have that (3) implies (1). Furthermore, if  $f \in C_c^\infty(\mathbb{R}_+)$  with  $\|f\|_{\mathcal{W}_p^\alpha} \leq 1$ ,  $\text{supp } f \subset [\frac{1}{2}, 2]$ ,  $n \in \mathbb{Z}$  and  $g = f(2^n \cdot)$ , then  $\|g\|_{\mathcal{W}_p^\alpha} \lesssim 1$ . Indeed, by the fixed support of  $f$ , the Sobolev norms of  $f(e^{(\cdot)+n \log(2)})$  and  $f((\cdot) + n \log(2))$  are equivalent, and we thus have

$$\begin{aligned} \|g\|_{\mathcal{W}_p^\alpha} &= \|f(2^n \cdot)\|_{\mathcal{W}_p^\alpha} = \|f(2^n e^{(\cdot)})\|_{\mathcal{W}_p^\alpha} \\ &= \|f(e^{(\cdot)+n \log(2)})\|_{\mathcal{W}_p^\alpha} \cong \|f((\cdot) + n \log(2))\|_{\mathcal{W}_p^\alpha} = \|f\|_{\mathcal{W}_p^\alpha} \leq 1. \end{aligned}$$

Thus, (1) implies (2). It remains to show that (2) implies (3). Consider a function  $\phi \in H_0^\infty(\Sigma_\nu)$  such that  $\sum_{n \in \mathbb{Z}} \phi^3(2^{-n} \lambda) = 1$  for any  $\lambda \in \Sigma_\nu$  and some  $\nu > \sigma$ . Furthermore, consider a function  $\eta \in C_c^\infty$  with  $\text{supp } \eta \subset [\frac{1}{2}, 2]$  such that  $\sum_{n \in \mathbb{Z}} \eta(2^{-n} t) = 1$  for any  $t > 0$ . Let  $f_1, \dots, f_N \in C^\infty(\mathbb{R}_+)$  with  $\|f_j\|_{\mathcal{H}_p^\alpha} \leq 1$  for  $j = 1, \dots, N$ . Then for  $x_j \in D_A$ , the calculus core,

$$\begin{aligned} \mathbb{E} \left\| \sum_{j=1}^N \epsilon_j f_j(A) x_j \right\| &= \mathbb{E} \left\| \sum_{j=1}^N \sum_{k \in \mathbb{Z}} \epsilon_j \eta(2^{-k} A) f_j(A) x_j \right\| \\ &\cong \mathbb{E} \mathbb{E}' \left\| \sum_{j=1}^N \sum_{n, k \in \mathbb{Z}} \epsilon_j \epsilon'_n \phi^2(2^{-n} A) \eta(2^{-k} A) f_j(A) x_j \right\| \\ &\leq \sum_{l \in \mathbb{Z}} \mathbb{E} \mathbb{E}' \left\| \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \epsilon_j \epsilon'_n [\phi(2^{-n} A) \eta(2^{-n-l} A) f_j(A)] \phi(2^{-n} A) x_j \right\| \\ &\leq \left( \sum_{l \in \mathbb{Z}} C_l \right) \mathbb{E} \mathbb{E}' \left\| \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \epsilon_j \epsilon'_n \phi(2^{-n} A) x_j \right\| \end{aligned}$$



$$\lesssim \left( \sum_{l \in \mathbb{Z}} C_l \right) \mathbb{E} \left\| \sum_{j=1}^N \epsilon_j x_j \right\|,$$

where we have used that  $\|x\| \cong \mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \epsilon_n \phi^2(2^{-n}A)x \right\| \cong \mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \epsilon_n \phi(2^{-n}A)x \right\|$ . Indeed, the second expression is estimated by the third one, since  $\{\phi(2^{-n}A) : n \in \mathbb{Z}\}$  is  $R$ -bounded by the  $R$ -boundedness of the  $H^\infty(\Sigma_\nu)$  calculus [43, 12.8 Theorem]. The third expression is estimated by the first one according to [43, 12.2 Theorem and 12.3 Remark]. Finally the first expression is estimated by the second one again by [43, 12.2 Theorem and 12.3 Remark] and  $|\langle x, x' \rangle| = |\mathbb{E} \langle \sum_{n \in \mathbb{Z}} \epsilon_n \phi^2(2^{-n}A)x, \sum_{k \in \mathbb{Z}} \epsilon_k \phi(2^{-k}A)x' \rangle| \leq \mathbb{E} \left\| \sum_n \epsilon_n \phi^2(2^{-n}A)x \right\| \mathbb{E} \left\| \sum_k \epsilon_k \phi(2^{-k}A)x' \right\| \lesssim \mathbb{E} \left\| \sum_n \epsilon_n \phi^2(2^{-n}A)x \right\| \|x'\|$ .

Furthermore, we used property  $(\alpha)$  in the fourth line, and  $C_l = R(\{\phi(2^{-n}A)\eta(2^{-n-l}A)f_j(A) : n \in \mathbb{Z}, j = 1, \dots, N\})$  and

$$\begin{aligned} C_l &\lesssim \sup_{j=1, \dots, N} \sup_{n \in \mathbb{Z}} \|\phi(2^l \cdot) \eta f_j(2^{n+l} \cdot)\|_{\mathcal{W}_p^\alpha} \\ &\lesssim \sup_{j=1, \dots, N} \sup_{k \in \mathbb{Z}} \|\eta f_j(2^k \cdot)\|_{\mathcal{W}_p^\alpha} \sup_{m=0, \dots, [\alpha]+1} \sup_{t \in [\frac{1}{2}, 2]} t^m \left| \frac{d^m}{dt^m} \phi(2^l \cdot)(t) \right| \\ &\lesssim \sup_{j=1, \dots, N} \|f_j\|_{\mathcal{H}_p^\alpha} 2^{-\epsilon|l|}, \\ &\leq 2^{-\epsilon|l|} \end{aligned}$$

where  $\epsilon > 0$  and we used the fact that  $\phi \in H_0^\infty(\Sigma_\nu)$ . Hence  $\sum_{l \in \mathbb{Z}} C_l \lesssim \sup_{j=1, \dots, N} \|f_j\|_{\mathcal{H}_p^\alpha} < \infty$ . We have shown that

$$(5.2) \quad \{f(A) : f \in C^\infty(\mathbb{R}_+), \|f\|_{\mathcal{H}_p^\alpha} \leq 1\}$$

is  $R$ -bounded. In particular, since  $C^\infty(\mathbb{R}_+) \supset \bigcap_{\omega > 0} H^\infty(\Sigma_\omega)$ ,  $A$  has a bounded  $\mathcal{H}_p^\alpha$  calculus in the sense of Definition 3.8, and by taking the closure of (5.2), this calculus is  $R$ -bounded.

If  $X$  does not have property  $(\alpha)$ , then repeat the proof of (2)  $\implies$  (3) above with a single function  $f \in C^\infty(\mathbb{R}_+)$ ,  $\|f\|_{\mathcal{H}_p^\alpha} \leq 1$  to get in a similar manner (2)  $\implies$  (3'). Now  $\{\phi(2^{-n}A) : n \in \mathbb{Z}\}$  is  $R$ -bounded since  $A$  is  $R$ -sectorial, as soon as  $\phi \in H_0^\infty(\Sigma_\theta)$  with  $\theta > \omega_R(A)$ .  $\square$

From the proof of Theorem 5.1, we obtain the following.

**Corollary 5.2.** Let  $A$  be a 0-sectorial operator on some Banach space  $X$ . Assume that  $A$  has an  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \pi)$  and an auxiliary calculus  $\Phi_A : \mathcal{H}_p^\alpha \rightarrow B(D(\theta), X)$  for some  $\theta > 0$ ,  $\alpha > \frac{1}{p}$  and  $1 \leq p < \infty$ . Let  $(\varphi_n)_{n \in \mathbb{Z}}$  be a dyadic partition of unity.

- (1) If for any  $f \in \mathcal{H}_p^\alpha$  the set  $\{(\varphi_n f)(A) : n \in \mathbb{Z}\}$  is  $R$ -bounded, then  $A$  has a bounded  $\mathcal{H}_p^\alpha$  calculus.
- (2) If  $X$  has property  $(\alpha)$ , then  $A$  has an  $R$ -bounded  $\mathcal{H}_p^\alpha$  calculus if and only if  $\{(\varphi_n f)(A) : n \in \mathbb{Z}\}$  is  $R$ -bounded for any  $f \in \mathcal{H}_p^\alpha$ .

*Proof.* For the second part, note that  $\|\varphi_n f\|_{\mathcal{H}_p^\alpha} \lesssim \|\varphi_n\|_{\mathcal{H}_p^\alpha} \|f\|_{\mathcal{H}_p^\alpha} \lesssim \|f\|_{\mathcal{H}_p^\alpha}$ , so that the “only if” part follows. To complete the proof, it now suffices to show that if  $\{(\varphi_n f)(A) : n \in \mathbb{Z}\}$  is  $R$ -bounded for any  $f \in \mathcal{H}_p^\alpha$ , then condition (5.1) from Theorem 5.1 is satisfied.

First note that it follows from the closed graph theorem together with the existence of the auxiliary calculus  $\Phi_A$  that

$$(5.3) \quad R(\{(\varphi_n f)(A) : n \in \mathbb{Z}\}) \leq C \|f\|_{\mathcal{H}_p^\alpha}.$$

We claim that (5.3) implies

$$(5.4) \quad R(\{(\varphi_n f_n)(A) : n \in \mathbb{Z}\}) \leq C' \sup_{n \in \mathbb{Z}} \|\varphi_n f_n\|_{\mathcal{W}_p^\alpha}.$$

From (5.4) it is easy to see that (5.1) follows. To prove the claim, we let  $f_n \in \mathcal{H}_p^\alpha$  and set  $f = \sum_{n \in 3\mathbb{Z}} \varphi_n f_n$ . Then  $\|f\|_{\mathcal{H}_p^\alpha} \lesssim \sup_{n \in \mathbb{Z}} \|\varphi_n f_n\|_{\mathcal{W}_p^\alpha}$ . Set  $\widetilde{\varphi}_n = \varphi_{n-1} + \varphi_n + \varphi_{n+1}$ , so that  $\widetilde{\varphi}_n \varphi_m = \delta_{nm} \varphi_n$  for any  $m \in 3\mathbb{Z}$ . It is clear that (5.3) implies that also  $R(\{(\widetilde{\varphi}_n f)(A) : n \in \mathbb{Z}\}) \lesssim \|f\|_{\mathcal{H}_p^\alpha}$ . Then, since  $\widetilde{\varphi}_n f = \varphi_n f_n$  for any  $n \in 3\mathbb{Z}$ , we obtain

$$\begin{aligned} R(\{(\varphi_n f_n)(A) : n \in 3\mathbb{Z}\}) &= R(\{(\widetilde{\varphi}_n f)(A) : n \in 3\mathbb{Z}\}) \\ &\leq R(\{(\widetilde{\varphi}_n f)(A) : n \in \mathbb{Z}\}) \\ &\lesssim \|f\|_{\mathcal{H}_p^\alpha} \lesssim \sup_{n \in \mathbb{Z}} \|\varphi_n f_n\|_{\mathcal{W}_p^\alpha}. \end{aligned}$$

Now apply the same argument to  $f = \sum_{n \in 3\mathbb{Z}+k} \varphi_n f_n$  with  $k = 1, 2$ , to deduce (5.4).  $\square$

## 6. IMAGINARY POWERS AND THE $\mathcal{H}_p^\alpha$ CALCULUS

In this section, we establish sufficient conditions for the  $\mathcal{H}_p^\alpha$  calculus based on polynomial growth estimates of imaginary powers.

**Theorem 6.1.** Let  $A$  be a 0-sectorial and  $R$ -sectorial operator on  $X$  having an  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \pi)$ . Let  $r \in (1, 2]$ ,  $\frac{1}{r} > \frac{1}{\text{type } X} - \frac{1}{\text{cotype } X}$ .

- (1) If  $\{(1 + |t|)^{-\alpha} A^{it} : t \in \mathbb{R}\}$  is semi- $R$ -bounded and  $X$  has property  $(\alpha)$ , then  $A$  has an  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus for  $\beta > \alpha + \frac{1}{2}$ .
- (2) If  $\{(1 + |t|)^{-\alpha} A^{it} : t \in \mathbb{R}\}$  is bounded, then  $A$  has a bounded  $\mathcal{H}_r^\beta$  calculus for  $\beta > \alpha + \frac{1}{r}$ . If  $X$  has in addition property  $(\alpha)$ , then this calculus is  $R$ -bounded.

Conversely, if  $A$  has an  $R$ -bounded  $\mathcal{H}_p^\beta$  functional calculus for some  $\beta$  and  $p$  such that  $\beta > \frac{1}{p}$ , then  $\{(1 + |t|)^{-\alpha} A^{it} : t \in \mathbb{R}\}$  is  $R$ -bounded for any  $\alpha > \beta$ .

**Remark 6.2.** Recall that an  $L^p(U)$  space,  $1 < p < \infty$ , has type  $\min(p, 2)$  and cotype  $\max(2, p)$ , so that in this case,  $\frac{1}{r} > \left| \frac{1}{2} - \frac{1}{p} \right|$ . An  $L^p(U)$  space always has property  $(\alpha)$ .

*Proof of Theorem 6.1.* For (1) (resp.(2)), by Theorem 5.1, it suffices to show that  $A$  has an  $R$ -bounded  $\mathcal{W}_2^\beta$  calculus (resp. an  $R$ -bounded  $\mathcal{W}_r^\beta$  calculus).

We now prove (1). To this end, we show that

$$(S_{\text{BIP}})_\beta$$

The function  $t \mapsto \langle t \rangle^{-\beta} A^{it} x$  belongs to  $\gamma(\mathbb{R}, X)$  with norm  $\lesssim \|x\|$  for  $x \in X$ .

Here,  $\gamma(\mathbb{R}, X)$  is the Gaussian space as defined for example in [51, Definition 3.7].  $(S_{\text{BIP}})_\beta$  will then imply by [45, Proposition 3.3], see also [29, Corollary 3.19], that

$$\left\{ \int_{\mathbb{R}} f(t) \langle t \rangle^{-\beta} A^{it} dt : \|f\|_{L^2(\mathbb{R})} \leq 1 \right\} \text{ is } R\text{-bounded.}$$

Since  $\|f\|_{\mathcal{W}_2^\beta} \cong \|\hat{f}(t)\langle t \rangle^\beta\|_{L^2}$ , this implies by Lemma 4.6 that  $A$  has a  $\mathcal{W}_2^\beta$  calculus and that this calculus is  $R$ -bounded. Thus, (1) follows from  $(S_{\text{BIP}})_\beta$ , which we show now to hold. By [61, Proposition 4.1], we have that

$$\begin{aligned} \|\langle t \rangle^{-\beta} A^{it} x\|_{\gamma(\mathbb{R}, X)} &= \|\langle t \rangle^{-(\beta-\alpha)} \langle t \rangle^{-\alpha} A^{it} x\|_\gamma \\ &\leq \|\langle t \rangle^{-(\beta-\alpha)}\|_{L^2(\mathbb{R})} R_s(\langle t \rangle^{-\alpha} A^{it} : t \in \mathbb{R}) \|x\| \\ &\lesssim \|x\|. \end{aligned}$$

Thus,  $(S_{\text{BIP}})_\beta$  follows.

We now prove (2). Write  $U(t) = A^{it}$ . Clearly,  $t \mapsto \langle t \rangle^{-\beta} \|U(t)\|$  is dominated by a function in  $L^r(\mathbb{R})$ . Indeed,  $\langle t \rangle^{-\beta} \|U(t)\| = \langle t \rangle^{-(\beta-\alpha)} (\langle t \rangle^{-\alpha} \|U(t)\|)$ , and the first factor is in  $L^r(\mathbb{R})$  by the choice of  $\beta$ , and the second factor is bounded by the assumption (2). In particular, by Lemma 4.6,  $A$  has a  $\mathcal{W}_r^\beta$  calculus, and for  $f \in \mathcal{W}_r^\beta$ , we have

$$f(A)x = \frac{1}{2\pi} \int_{\mathbb{R}} \langle t \rangle^\beta (\hat{f}_e)(t) \langle t \rangle^{-\beta} U(t) x dt \quad (x \in X).$$

Since  $r \leq 2$ , we have by the Hausdorff-Young inequality  $\|\hat{f}_e(t) \langle t \rangle^\beta\|_{L^{r'}(\mathbb{R})} \lesssim \|f_e\|_{\mathcal{W}_r^\beta} = \|f\|_{\mathcal{W}_r^\beta}$ . Further, by the assumption  $\frac{1}{r} > \frac{1}{\text{type } X} - \frac{1}{\text{cotype } X}$ , we can apply Proposition 2.5 and consequently,

$$R(\{f(A) : \|f\|_{\mathcal{W}_r^\beta} \leq 1\}) \lesssim R(\{f(A) : \|\hat{f}_e(t) \langle t \rangle^\beta\|_{L^{r'}(\mathbb{R})} \leq 1\}) < \infty.$$

The converse statement follows by applying Lemmas 3.9 and 3.2.  $\square$

**Remark 6.3.** The fact that  $A$  has a Hörmander calculus provided that imaginary powers of  $A$  grow at most polynomially has been studied before by Meda [48]. He assumes that  $-A$  is self-adjoint and generates a contraction semigroup on  $L^p$  for all  $1 < p < \infty$ . Note that such an operator has an  $H^\infty$  calculus on  $L^p$  for any  $1 < p < \infty$  [12]. If furthermore the imaginary powers satisfy  $\|A^{it}\| \leq C_p(1 + |t|)^{\beta|\frac{1}{p}-\frac{1}{2}|}$  for all  $1 < p < \infty$  and some  $\beta > 0$ , then Meda shows that  $A$  has a  $\mathcal{H}_\infty^{\beta/2+1+\epsilon}$  calculus on  $L^p$  for  $1 < p < \infty$  [48, Corollary 1, Theorem 4]. By comparison our result guarantees a  $\mathcal{H}_r^{\beta|\frac{1}{p}-\frac{1}{2}|+\frac{1}{r}+\epsilon}$  calculus with  $\frac{1}{r} = |\frac{1}{p} - \frac{1}{2}| < 1$ , which is a stronger result according to Lemma 3.2 (2). Moreover, our functional calculus is  $R$ -bounded, not just bounded.

The optimality of the assumptions of Theorem 6.1 will be discussed in Section 8.

## 7. SEMIGROUPS AND THE $\mathcal{H}_p^\alpha$ CALCULUS

We have the following analogue of Theorem 6.1 considering  $R$ -bounds on the analytic semigroup of  $A$  in place of the imaginary powers.

**Theorem 7.1.** Let  $A$  be a 0-sectorial operator on  $X$  having an  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \pi)$ . Let  $r \in (1, 2]$  with  $\frac{1}{r} > \frac{1}{\text{type } X} - \frac{1}{\text{cotype } X}$  and  $\alpha \geq 0$ . (See Remark 6.2 for the  $L^q$ -case.)

(1) If

$$(7.1) \quad \left\{ \left( \frac{\pi}{2} - |\theta| \right)^\alpha \exp(-e^{i\theta} t A) : t > 0, \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right\} \text{ is } R\text{-bounded}$$

and  $X$  has property  $(\alpha)$ , then  $A$  has an  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus for  $\beta > \alpha + \frac{1}{2}$ .

(2) If

$$(7.2) \quad R\left(\left\{\exp(-e^{i\theta}t2^n A) : n \in \mathbb{Z}\right\}\right) \leq C\left(\frac{\pi}{2} - |\theta|\right)^{-\alpha}$$

with a constant  $C < \infty$  uniformly in  $t > 0$  and  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , then  $A$  has a bounded  $\mathcal{H}_r^\beta$  calculus for  $\beta > \alpha + \frac{1}{r}$ . If  $X$  has in addition property  $(\alpha)$ , then this calculus is  $R$ -bounded.

Conversely, if  $A$  has an  $R$ -bounded  $\mathcal{H}_p^\beta$  functional calculus for some  $p \in [1, \infty)$  and  $\beta > \frac{1}{p}$ , then  $\{(\frac{\pi}{2} - |\theta|)^\beta \exp(-e^{i\theta}tA) : t > 0, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$  is  $R$ -bounded.

*Proof.* For part (1), we can assume w.l.o.g. that  $\sigma < \frac{\pi}{2}$ . Indeed, the proof of (2) below which has weaker assumptions than (1) shows that  $A$  has a bounded  $\mathcal{H}_r^\beta$  calculus for  $\beta > \alpha + \frac{1}{r}$  and thus a bounded  $H^\infty(\Sigma_\sigma)$  calculus to any angle  $\sigma > 0$ . We show that the assumptions imply

$$(S_T)_{\alpha+\frac{1}{2}} \quad \text{The function } t \mapsto A^{1/2}T(e^{i\theta}t)x \text{ belongs to } \gamma(\mathbb{R}_+, dt, X) \\ \text{with norm } \lesssim \left(\frac{\pi}{2} - |\theta|\right)^{-\alpha-\frac{1}{2}}\|x\| \text{ for } x \in X \text{ and } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Here,  $\gamma(\mathbb{R}_+, dt, X)$  is the Gaussian space as defined for example in [34], [51, Definition 3.7]. Then  $(S_T)_{\alpha+\frac{1}{2}}$  will imply by [45, Proposition 3.3] and [29, Corollary 3.19] that

$$\left\{\int_0^\infty f(t)A^{\frac{1}{2}}T(e^{i\theta}t)dt : \|f\|_{L^2(\mathbb{R}_+, dt)} \leq 1\right\} \text{ is } R\text{-bounded with } R\text{-bound } \lesssim \left(\frac{\pi}{2} - |\theta|\right)^{-\alpha-\frac{1}{2}}$$

for  $|\theta| < \frac{\pi}{2}$ , which implies by [39] that  $A$  has an  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus for any  $\beta > \alpha + \frac{1}{2}$ .

By [34, Theorem 7.2, Proposition 7.7], the fact that  $A$  has an  $H^\infty(\Sigma_\sigma)$  calculus with angle  $\sigma < \frac{\pi}{2}$  implies that for  $x \in D(A) \cap R(A)$ ,  $\|A^{\frac{1}{2}}T(t)x\|_{\gamma(\mathbb{R}_+, X)} \lesssim \|x\|$ , so that by [51, Corollary 6.3] with the isomorphic mapping  $L^2(\mathbb{R}_+, dt) \rightarrow L^2(\mathbb{R}_+, dt)$ ,  $t \mapsto ts$

$$\|A^{\frac{1}{2}}T(ts)x\|_{\gamma(\mathbb{R}_+, X)} \lesssim s^{-\frac{1}{2}}\|x\|$$

for  $s > 0$ . Decompose

$$e^{\pm i(\frac{\pi}{2}-\omega)} = re^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})} + s$$

where  $s, r > 0$  are uniquely determined. By the law of sines,  $s \cong \omega$  for  $\omega \rightarrow 0 +$ . Then

$$A^{\frac{1}{2}}T(te^{\pm i(\frac{\pi}{2}-\omega)}) = T(tre^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})}) \circ A^{\frac{1}{2}}T(ts).$$

Therefore, by assumption of part (1),

$$\begin{aligned} \|A^{\frac{1}{2}}T(te^{\pm i(\frac{\pi}{2}-\omega)})x\|_{\gamma(\mathbb{R}_+, X)} &\leq R(\{T(tre^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})}) : t > 0\})\|A^{\frac{1}{2}}T(ts)x\|_{\gamma(\mathbb{R}_+, X)} \\ &\lesssim \omega^{-\alpha}\|A^{\frac{1}{2}}T(ts)x\|_{\gamma(\mathbb{R}_+, X)} \\ &\lesssim \omega^{-\alpha}\omega^{-\frac{1}{2}}\|A^{\frac{1}{2}}T(t)x\|_{\gamma(\mathbb{R}_+, X)} \\ &\lesssim \omega^{-\alpha-\frac{1}{2}}\|x\|. \end{aligned}$$

Now  $(S_T)_{\alpha+\frac{1}{2}}$  follows and part (1) is proved.

We now turn to part (2). Note that (7.2) implies that  $A$  is  $R$ -sectorial [43, 2.16 Example and 2.20 Theorem]. Then similarly to the proof of Theorem 6.1, (2) follows from Theorem 5.1, if we can show that (5.1) holds with  $\alpha$  replaced by  $\beta$  and  $p$  replaced by  $r$ . To this end, let  $f \in C_c^\infty(\mathbb{R}_+)$  with  $\text{supp } f \subset [\frac{1}{2}, 2]$  and  $\|f\|_{W_r^\beta} \leq 1$ . Write  $\delta = \beta - \alpha > \frac{1}{r}$ . The assumptions imply by an inspection of the proof of

[43, End of 2.16 Example] that the  $R$ -boundedness assumption of (1) hold for some larger  $\alpha'$ . This in turn implies by (a simplified version of) the proof of part (1) (see also Remark 7.2) that  $A$  has a  $\mathcal{W}_p^{\alpha'}$  calculus. Then by Lemma 4.6 (3), for  $x \in D_A$  and  $n \in \mathbb{Z}$ ,

$$f(2^n A)x = \frac{1}{2\pi} \int_{\mathbb{R}} \check{f}(s) \exp(-is2^n A)x ds = \frac{1}{2\pi} \int_{\mathbb{R}} \check{f}(s-i) \exp((-is-1)2^n A)x ds.$$

Here we performed a shift of a complex contour integral, which is allowed, since  $|\check{f}(s-it)|$  decays faster than any polynomial for  $|s| \rightarrow \infty$  and  $t \in [0, 1]$ , and  $\|\exp((-is-1)2^n A)x\|$  grows only polynomially as  $|s| \rightarrow \infty$ . We decompose the integrand as

$$(7.3) \quad \left[ \frac{1}{|-is-1|^{\alpha+\delta}} \exp((-is-1)2^n A)x \right] \left[ \check{f}(s-i) |-is-1|^{\alpha+\delta} \right].$$

Consider the first bracket as a function in the variable  $s \in \mathbb{R}$ , with values in  $B(\text{Rad}(X))$ , where  $\text{Rad}(X)$  is the closed subspace of  $L^2(\Omega, X)$  which is generated by elements of the form  $\epsilon_n \otimes x$  with  $x \in X$  and  $(\epsilon_n)_{n \in \mathbb{Z}}$  a sequence of independent Rademacher variables over the probability space  $\Omega$  (cf. Section 2). This means that  $\sum_{n \in \mathbb{Z}} \epsilon_n \otimes x_n \mapsto \sum_{n \in \mathbb{Z}} \epsilon_n \otimes \frac{1}{|-is-1|^{\alpha+\delta}} \exp((-is-1)2^n A)x_n$ . As  $|-is-1| \cong (\frac{\pi}{2} - |\theta|)^{-1}$  for the choice  $s = \tan \theta$  and  $t^2 = 1 + s^2$ , so that  $e^{i\theta}t = 1 + is$ , one sees that the assumption in (2) implies that  $\left\{ \frac{1}{|-is-1|^\alpha} \exp((-is-1)2^n A) : n \in \mathbb{Z} \right\}$  is  $R$ -bounded over  $X$  with uniform  $R$ -bound in  $s \in \mathbb{R}$ . Thus,

$$\left\| \frac{1}{|-is-1|^\delta} \left\| \left( \frac{1}{|-is-1|^\alpha} \exp((-is-1)2^n A) \right)_{n \in \mathbb{Z}} \right\|_{B(\text{Rad}(X))} \right\|_{L^r(\mathbb{R}, ds)} \lesssim \left\| \frac{1}{|-is-1|^\delta} \right\|_{L^r(\mathbb{R}, ds)},$$

which is finite by the choice  $\delta r > 1$ . The  $L^{r'}(\mathbb{R}, ds)$  norm of the expression in the second bracket of (7.3) is estimated by

$$\|\check{f}(s-i) |-is-1|^{\alpha+\delta}\|_{L^{r'}(\mathbb{R}, ds)} = \|[f \exp]^\vee(s) |-is-1|^{\alpha+\delta}\|_{L^{r'}} \lesssim \|f \exp\|_{W_r^\beta} \lesssim \|f\|_{W_r^\beta},$$

where the last estimate follows from  $\text{supp } f \subset [\frac{1}{2}, 2]$ . Note that  $\text{Rad}(X)$  has the same type and cotype as  $X$  as a closed subspace of  $L^2(\Omega, X)$ . By Proposition 2.5, it follows that

$$\left\{ (f(2^n A))_{n \in \mathbb{Z}} : f \in C_c^\infty, \text{supp } f \subset [\tfrac{1}{2}, 2], \|f\|_{W_r^\beta} \leq 1 \right\} \text{ is } R\text{-bounded over } \text{Rad}(X).$$

This implies that also

$$(7.4) \quad \left\{ f(2^n A) : f \in C_c^\infty, \text{supp } f \subset [\tfrac{1}{2}, 2], \|f\|_{W_r^\beta} \leq 1, n \in \mathbb{Z} \right\} \text{ is } R\text{-bounded over } X.$$

Indeed, let  $x_i \in X$ ,  $n_i \in \mathbb{Z}$  and  $f_i \in C_c^\infty$  such that  $\text{supp } f_i \subset [\frac{1}{2}, 2]$  and  $\|f_i\|_{W_r^\beta} \leq 1$ . Put  $y_i = \epsilon_{n_i} \otimes x_i \in \text{Rad}(X)$ . Then, with  $(\epsilon'_i)_i$  being another sequence of independent Rademachers over a different probability space  $\Omega'$ , we have

$$\begin{aligned} \mathbb{E}' \left\| \sum_i \epsilon'_i f_i(2^{n_i} A)x_i \right\| &= \mathbb{E} \mathbb{E}' \left\| \sum_i \epsilon'_i \epsilon_{n_i} f_i(2^{n_i} A)x_i \right\| = \mathbb{E} \mathbb{E}' \left\| \sum_i \epsilon'_i (f_i(2^{n_i} A))_{n \in \mathbb{Z}}(y_i) \right\| \\ &\lesssim \mathbb{E} \mathbb{E}' \left\| \sum_i \epsilon'_i y_i \right\| = \mathbb{E}' \left\| \sum_i \epsilon'_i x_i \right\|. \end{aligned}$$

This shows (7.4), and thus (5.1).

The converse statement follows by applying Lemma 3.9.  $\square$

**Remark 7.2.** In Theorem 7.1, the Hörmander functional calculus is a consequence of  $R$ -boundedness conditions on the analytic semigroup generated by  $-A$ . If one weakens the assumptions to norm bound conditions as in the following, one gets by the same technique a weaker Sobolev functional calculus, but not necessarily a  $\mathcal{H}_p^\alpha$  calculus, cf. Subsection 8.3.

Let  $A$  be a 0-sectorial operator on some Banach space  $X$ . Assume that for some  $\alpha > 0$ ,

$$\|\exp(-zA)\| \leq C \left( \frac{|z|}{\operatorname{Re} z} \right)^\alpha \quad (z \in \mathbb{C}_+).$$

Then for each  $R > 0$ , the set  $\{f(A) : f \in W_r^\beta, \operatorname{supp} f \subset [0, R], \|f\|_{W_r^\beta} \leq 1\}$  is  $R$ -bounded, where  $r \leq 2$ ,  $\frac{1}{r} > \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$  and  $\beta > \alpha + \frac{1}{r}$ . Furthermore,  $A$  has an auxiliary calculus  $\Phi_A : \mathcal{H}_r^\beta \rightarrow B(D(\theta), X)$  for any  $\theta > 0$ .

*Proof.* Let  $\delta > \frac{1}{r} > \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$  and  $\beta = \alpha + \delta$ . Clearly the assumptions imply that  $\{|1 + is|^{-\alpha} \exp(-(1 + is)A) : s \in \mathbb{R}\}$  is bounded. Then as in the proof of Theorem 7.1, we have for  $f \in W_r^\beta$

$$\begin{aligned} f(A)x &= \frac{1}{2\pi} \int_{\mathbb{R}} \check{f}(s - i) \exp(-(is + 1)A) x ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} [\check{f}(s - i) \langle s \rangle^\beta] [\langle s \rangle^{-\delta} \langle s \rangle^{-\alpha} \exp(-(is + 1)A) x] ds. \end{aligned}$$

The second bracket is in  $L^r$  by the assumption  $\delta r > 1$ . The first bracket is in  $L^{r'}(\mathbb{R})$  with  $\|\check{f}(s - i) \langle s \rangle^{\alpha + \delta}\|_{L^{r'}} \leq \|f \exp\|_{W_r^\beta} \lesssim \|f\|_{W_r^\beta}$  by the Hausdorff-Young inequality, as soon as  $f$  has support in  $[0, R]$ . Then the first statement follows with Proposition 2.5.

For the second statement, note that the first part shows  $\|f(A)\| \lesssim \|f\|_{W_r^\beta} \cong \|f\|_{\mathcal{W}_r^\beta}$  for any  $f$  with  $\operatorname{supp} f \subset [\frac{1}{2}, 2]$ . Since  $2^{-n}A$  satisfies the same assumptions as  $A$  for any  $n \in \mathbb{Z}$ , we deduce  $\|f(2^{-n}A)\| \lesssim \|f\|_{\mathcal{W}_r^\beta}$  for any  $f$  with  $\operatorname{supp} f \subset [\frac{1}{2}, 2]$ , uniformly in  $n \in \mathbb{Z}$ . Now we take some dyadic partition of unity  $(\varphi_n)_{n \in \mathbb{Z}}$ ,  $\theta > 0$  and recall  $\rho(\lambda) = \lambda/(1 + \lambda)^2$ . We have

$$\begin{aligned} \|(\rho^\theta f)(A)\| &= \left\| \sum_{n \in \mathbb{Z}} (\varphi_n \rho^\theta f)(A) \right\| = \left\| \sum_{n \in \mathbb{Z}} [\varphi_0 \rho^\theta(2^n \cdot) f(2^n \cdot)](2^{-n}A) \right\| \\ &\lesssim \sum_{n \in \mathbb{Z}} \|\varphi_0 \rho^\theta(2^n \cdot) f(2^n \cdot)\|_{\mathcal{W}_r^\beta} \lesssim \sum_{n \in \mathbb{Z}} \|\varphi_0 \rho^{\theta/2}(2^n \cdot)\|_{\mathcal{W}_r^\beta} \|\rho^{\theta/2}(2^n \cdot) f(2^n \cdot)\|_{\mathcal{W}_r^\beta} \\ &\lesssim \sum_{n \in \mathbb{Z}} 2^{-|n|\theta/2} \|\rho^{\theta/2} f\|_{\mathcal{W}_r^\beta} \lesssim \|f\|_{\mathcal{H}_r^\beta}, \end{aligned}$$

where the last but one estimate follows similarly to the proof of Theorem 5.1, and the last estimate follows from the argument in the proof of Lemma 4.3. Thus, the auxiliary calculus  $\Phi_A : \mathcal{H}_r^\beta \rightarrow B(D(\theta), X)$  is bounded.  $\square$

**Remark 7.3.** The connection of the exponent of the  $\mathcal{H}_\infty^\alpha$  calculus with the growth rate of the analytic semigroup when approaching the imaginary axis as in condition (1) or (2) of Theorem 7.1 has been studied already by Duong in [20, Section 3]. There, the calculus of a Laplacian operator on a Nilpotent Lie group is investigated. A Hörmander calculus is obtained from a kernel estimate of the analytic semigroup. Note that in Duong's situation, Gaussian estimates for the semigroup are at hand.

For a comparison of Theorem 7.1 with spectral multiplier theorems for Gaussian estimates, we refer to Subsection 8.2.

Next we compare conditions on the wave operator associated with  $A$ , i.e. (variants of) the boundary value on the imaginary axis of the analytic semigroup generated by  $-A$ , with conditions on the analytic semigroup. Consider the following assertions.

$$(7.5) \quad \left\{ (1 + |s|2^n A)^{-\alpha} e^{is2^n A} : n \in \mathbb{Z} \right\} \text{ is } R\text{-bounded uniformly in } s \in \mathbb{R}.$$

$$(7.6) \quad \left\{ (1 + |s|A)^{-\alpha} e^{isA} : s \in \mathbb{R} \right\} \text{ is } R\text{-bounded}.$$

Note that these assertions include that the operators in question are defined on  $X$  and bounded. They are well-defined operators at least on the domain  $D_0 = R(e^{-A})$ , (which is dense in  $X$  by the analyticity of the dual semigroup  $(e^{-tA})'$ ), by the formula  $(1 + |s|A)^{-\alpha} e^{isA} x = (1 + |s|A)^{-\alpha} e^{(is-1)A} y$  for  $x = e^{-A} y \in D_0$ . Operators as in 7.6 are considered in [16], where they are called regularized semigroup, and in particular in [52, Sections 7.3, 7.4.2], [49, Theorems 2,3] in connection with spectral multipliers. Moreover, the link with analytic semigroups on the right half plane is studied in [9, Theorems 2.2, 2.3]. Clearly, (7.6) implies (7.5).

We have the following sufficient conditions for the  $\mathcal{H}_r^\beta$  calculus in terms of the wave operators.

**Theorem 7.4.** Let  $A$  be a 0-sectorial operator on a Banach space  $X$  with property  $(\alpha)$  having an  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \frac{\pi}{2})$ . Let  $r \in (1, 2]$ ,  $\frac{1}{r} > \frac{1}{\text{type } X} - \frac{1}{\text{cotype } X}$ . (See Remark 6.2 for the  $L^q$ -case.)

- (1) If  $\{(1 + |s|A)^{-\alpha} \exp(isA) : s \in \mathbb{R}\}$  is  $R$ -bounded, then  $A$  has an  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus for  $\beta > \alpha + \frac{1}{2}$ .
- (2) If  $\{(1 + |s|2^n A)^{-\alpha} \exp(is2^n A) : n \in \mathbb{Z}\}$  is  $R$ -bounded uniformly in  $s \in \mathbb{R}$ , then  $A$  has an  $R$ -bounded  $\mathcal{H}_r^\beta$  calculus for  $\beta > \alpha + \frac{1}{r}$ .

Conversely, if  $A$  has an  $R$ -bounded  $\mathcal{H}_p^\beta$  functional calculus for some  $\beta$  and  $p \in [1, \infty)$  such that  $\beta > \frac{1}{p}$ , then  $\{(1 + |s|A)^{-\beta} \exp(isA) : s \in \mathbb{R}\}$  is  $R$ -bounded.

*Proof.* By Proposition 7.5 below, the assumptions in (1) (resp. (2)) imply the assumptions in (1) (resp. (2)) of Theorem 7.1, so that (1) and (2) above follow immediately. For the converse statement, we refer again to Lemma 3.9.  $\square$

**Proposition 7.5.** Let  $A$  have a bounded  $H^\infty(\Sigma_\sigma)$  calculus for some  $\sigma \in (0, \frac{\pi}{2})$ . Let the underlying Banach space have property  $(\alpha)$ . Then for  $\alpha > 0$ , we have

$$(7.6) \implies (7.1) \text{ and } (7.5) \implies (7.2).$$

*Proof.* (7.6)  $\implies$  (7.1): Note first that for any  $\omega \in (\sigma, \frac{\pi}{2})$ , we have

$$(7.7) \quad \{f(A) : \|f\|_{\infty, \omega} \leq 1\} \text{ is } R\text{-bounded}.$$

Indeed, since  $X$  has property  $(\alpha)$ , by [43, Theorem 12.8], (7.7) follows. In particular,

$$\left\{ \left(\frac{\pi}{2} - |\theta|\right)^\alpha T(e^{i\theta} t) : t > 0, |\theta| \leq \frac{\pi}{2} - \omega \right\} \text{ is } R\text{-bounded}.$$

Fix some  $\omega \in (\sigma, \frac{\pi}{2})$  for the rest of the proof. Thus it remains to show that

$$(7.8) \quad \left\{ \left(\frac{\pi}{2} - |\theta|\right)^\alpha T(e^{i\theta} t) : t > 0, |\theta| \in \left(\frac{\pi}{2} - \omega, \frac{\pi}{2}\right) \right\} \text{ is } R\text{-bounded}.$$

We write  $e^{i\theta}t = r + is$  with real  $r$  and  $s$ . Then for  $x = e^{-A}y \in D_0$ , we have

$$\begin{aligned} \left(\frac{r}{|s|}\right)^\alpha T(r + is)x &= \left(\frac{r}{|s|}\right)^\alpha T(r + 1 + is)y \\ &= \left[(1 + |s|A)^{-\alpha} e^{(-is-1)A}\right] \circ \left[\left(\frac{r}{|s|}\right)^\alpha (1 + rA)^{-\alpha} (1 + |s|A)^\alpha\right] \circ [(1 + rA)^\alpha T(r)] y \\ &= \left[(1 + |s|A)^{-\alpha} e^{-isA}\right] \circ \left[\left(\frac{r}{|s|}\right)^\alpha (1 + rA)^{-\alpha} (1 + |s|A)^\alpha\right] \circ [(1 + rA)^\alpha T(r)] x. \end{aligned}$$

We show that all three brackets form  $R$ -bounded sets for  $r + is$  varying in  $\{z \in \mathbb{C} \setminus \{0\} : |\arg z| \in (\frac{\pi}{2} - \omega, \frac{\pi}{2})\}$ . Note that for  $|\theta| \in (\frac{\pi}{2} - \omega, \frac{\pi}{2})$ , we have  $\frac{\pi}{2} - |\theta| \cong \frac{r}{|s|}$ , so that this will imply (7.8) by Kahane's contraction principle. The assumption (7.6) implies that the first bracket is  $R$ -bounded with  $s$  varying in  $\mathbb{R}$ . We show in a moment that

$$(7.9) \quad \left(\frac{r}{|s|}\right)^\alpha (1 + r(\cdot))^{-\alpha} (1 + |s|(\cdot))^\alpha \text{ is uniformly bounded in } H^\infty(\Sigma_{\frac{\pi}{2}}).$$

Then the fact that the second bracket is  $R$ -bounded follows from (7.7). For  $\lambda \in \Sigma_{\frac{\pi}{2}}$ ,

$$\begin{aligned} \left(\frac{r}{|s|}\right)^\alpha |(1 + r\lambda)^{-\alpha} (1 + |s|\lambda)^\alpha| &= \left|\frac{\frac{1}{|s|} + \lambda}{\frac{1}{r} + \lambda}\right|^\alpha \\ &\cong \left(\frac{\frac{1}{|s|} + |\lambda|}{\frac{1}{r} + |\lambda|}\right)^\alpha \\ &\lesssim 1, \end{aligned}$$

since  $|s| \gtrsim r$  by the restriction  $|\theta| \in (\frac{\pi}{2} - \omega, \frac{\pi}{2})$ . Thus, (7.9) follows. Finally, (7.7) with  $f(\lambda) = (1 + \lambda)^\alpha e^{-\lambda}$  implies that the third bracket is  $R$ -bounded with  $r$  varying in  $(0, \infty)$ . Now (7.1) follows since  $x$  is from the dense subspace  $D_0$ .

(7.5)  $\implies$  (7.2) : The proof is similar to (7.6)  $\implies$  (7.1).  $\square$

**Remark 7.6.** If  $A$  satisfies all the assumptions of Theorem 7.1 or 7.4, parts (1) or (2), but is not injective, then one has the following variants for a reflexive Banach space  $X$ . The semigroup  $\exp(-zA)$  leaves  $N(A)$  and  $\overline{R(A)}$  invariant and  $\exp(-zA)|_{\overline{R(A)}}$  is again an analytic semigroup with generator  $A_1$ , where  $A = A_1 \oplus 0$  is the decomposition on  $\overline{R(A)} \oplus N(A)$  from Subsection 3.1. Moreover, this semigroup satisfies the conditions of (1) or (2) on the space  $\overline{R(A)}$  in Theorem 7.1. Thus also the conclusions of (1) and (2) hold for  $A_1$  and  $A$ . A similar remark holds for (1) and (2) of Theorem 7.4.

**Remark 7.7.** We end this section with a sufficient condition for the Hörmander calculus in terms of an  $R$ -boundedness condition on resolvents. It is not optimal: we loose one order in the differentiation parameter instead of  $\frac{1}{2}$  when passing from  $R$ -bounded resolvents to the functional calculus.

Let  $A$  be a 0-sectorial operator with  $H^\infty$  calculus on a space with property  $(\alpha)$ . Consider the following condition.

$$(7.10) \quad R\left(\{tR(e^{i\theta}t, A) : t > 0\}\right) \lesssim |\theta|^{-\alpha_1} \quad (0 < |\theta| < \pi).$$



Then (7.10) implies that  $A$  has an  $R$ -bounded  $\mathcal{H}_2^{\alpha_1}$  calculus. Conversely, an  $R$ -bounded  $\mathcal{H}_2^\alpha$  calculus of  $A$  implies that (7.10) holds with  $\alpha_1 = \alpha + 1$ .

*Proof.* Using the resolvent identity one can prove in a similar way as in Theorem 6.1 that (7.10)  $\implies$  (7.11) with  $\alpha_1 = \alpha_2$ , and

$$(7.11) \quad \|A^{\frac{1}{2}}R(e^{i\theta}t, A)x\|_{\gamma(\mathbb{R}_+, dt, X)} \lesssim |\theta|^{-\alpha_2}\|x\| \quad (0 < |\theta| < \pi).$$

Further, (7.11) implies that  $A$  has an  $R$ -bounded  $\mathcal{H}_2^{\alpha_2}$  calculus. For this fact and the definition of  $\gamma(\mathbb{R}_+, dt, X)$ , we refer to [36]. The converse statement follows from a Hörmander norm estimate of  $t(e^{i\theta}t - (\cdot))^{-1}$ .  $\square$

## 8. EXAMPLES AND COUNTEREXAMPLES

**8.1. Comparison with the Hörmander theorem for the Laplace operator on  $\mathbb{R}^n$ .** It is known that the classical Hörmander spectral multiplier theorem for the Laplace operator on  $L^p(\mathbb{R}^d)$ ,  $1 < p < \infty$ , can be improved as follows (see e.g. [10, Theorem A, p. 7], [57]).  $-\Delta$  has a  $\mathcal{H}_2^\beta$  calculus on  $L^p(\mathbb{R}^d)$  with

$$\beta > \beta_p := \max\left(\frac{d}{d+1}, d\left|\frac{1}{p} - \frac{1}{2}\right|\right).$$

How close can we come to this result with our general method? By the well known  $R$ -boundedness of the Gaussian kernel (see e.g. [43, Section 5.4]) we get

$$(8.1) \quad R\left(\left\{\exp(-e^{i\theta}2^n t A) : n \in \mathbb{Z}\right\}\right) \leq C\left(\frac{\pi}{2} - |\theta|\right)^{-\alpha} \quad \left(t > 0, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

for  $A = -\Delta$  and all  $p \in (1, \infty)$ , and  $\alpha = \frac{d}{2}$ . By complex Stein interpolation between  $p = 2$  (where  $\alpha = 0$ ) and  $p$  close to 1 or  $\infty$ , we can improve the exponent in (8.1) to all  $\alpha$  with

$$\alpha > \alpha_p = d\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Then by Theorem 7.1,  $-\Delta$  has a  $\mathcal{H}_2^\beta$  calculus on  $L^p(\mathbb{R}^d)$  of order  $\beta > \alpha_p + \frac{1}{2}$ . Hence our theorem, when applied to  $-\Delta$  directly, overestimates the necessary smoothness order for  $\beta_p > \frac{d}{d+1}$  by

$$\alpha_p + \frac{1}{2} - \beta_p = \frac{1}{2}.$$

This gap can be narrowed by considering  $A = (-\Delta)^{\frac{1}{2}}$  instead of  $-\Delta$ . Since  $A^2 = -\Delta$ , both operators have a Hörmander calculus of the same order. But the Poisson semigroup generated by  $A$  has an  $R$ -bound (see [37, Section 4]) which gives (8.1) with  $\alpha = \frac{d-1}{2}$  for all  $p \in (1, \infty)$ . Again by complex Stein interpolation, we get (8.1) for any exponent larger than

$$\alpha'_p = (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Theorem 7.1 guarantees now a  $\mathcal{H}_2^\alpha$  calculus for  $A$  of order  $\alpha'_p + \frac{1}{2}$ . Hence the gap  $\alpha'_p + \frac{1}{2} - \beta_p = \frac{1}{2} - \left|\frac{1}{p} - \frac{1}{2}\right|$ , disappears for  $p \rightarrow \infty$  or  $p \rightarrow 1$ . In particular, Theorem 7.1 implies a  $\mathcal{H}_2^{d/2}$ -calculus for the Laplace operator on  $L^p(\mathbb{R}^d)$  for all  $1 < p < \infty$ . The sharp result with  $\beta > \beta_p$  mentioned in the beginning requires more advanced methods, e.g. see [10]. For  $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{d+1}$ , the optimal order of the Hörmander calculus still seems to be unknown.

**8.2. (Generalized) Gaussian estimates imply the  $R$ -boundedness of the semigroup.** The considerations of Subsection 8.1 can be extended to (generalized) Gaussian estimates. We compare them with the  $R$ -boundedness condition of the semigroup as in Theorem 7.1, and compare our spectral multiplier theorem with the ones in the literature.

**Definition 8.1.** Let  $\Omega$  be a topological space which is equipped with a distance  $\rho$  and a Borel measure  $\mu$ . Let  $d \geq 1$  be an integer.  $\Omega$  is called a homogeneous space of dimension  $d$  if there exists  $C > 0$  such that for any  $x \in \Omega$ ,  $r > 0$  and  $\lambda \geq 1$  :

$$\mu(B(x, \lambda r)) \leq C \lambda^d \mu(B(x, r)).$$

Typical cases of homogeneous spaces are open subsets of  $\mathbb{R}^d$  with Lipschitz boundary and Lie groups with polynomial volume growth, in particular stratified nilpotent Lie groups (see e.g. [24]).

We will consider operators satisfying the following assumption.

**Assumption 8.2.**  $A$  is a self-adjoint positive (injective) operator on  $L^2(\Omega)$ , where  $\Omega$  is a homogeneous space of a certain dimension  $d$ . Further, there exists some  $p_0 \in [1, 2)$  such that the semigroup generated by  $-A$  satisfies the so-called generalized Gaussian estimate (see e.g. [3, (GGE)]):

(GGE)

$$\|\chi_{B(x, r_t)} e^{-tA} \chi_{B(y, r_t)}\|_{p_0 \rightarrow p'_0} \leq C \mu(B(x, r_t))^{\frac{1}{p'_0} - \frac{1}{p_0}} \exp\left(-c(\rho(x, y)/r_t)^{\frac{m}{m-1}}\right) \quad (x, y \in \Omega, t > 0).$$

Here,  $p'_0$  is the conjugated exponent to  $p_0$ ,  $C, c > 0$ ,  $m \geq 2$  and  $r_t = t^{\frac{1}{m}}$ ,  $\chi_B$  denotes the characteristic function of a  $B$ , where  $B(x, r) = \{y \in \Omega : \rho(y, x) < r\}$  and  $\|\chi_{B_1} T \chi_{B_2}\|_{p_0 \rightarrow p'_0} = \sup_{\|f\|_{p_0} \leq 1} \|\chi_{B_1} \cdot T(\chi_{B_2} f)\|_{p'_0}$ .

If  $p_0 = 1$ , then it is observed in [6] that (GGE) is equivalent to the usual Gaussian estimate, i.e.  $e^{-tA}$  has an integral kernel  $k_t(x, y)$  satisfying the pointwise estimate (cf. e.g. [21, Assumption 2.2])

$$(GE) \quad |k_t(x, y)| \lesssim \mu(B(x, t^{\frac{1}{m}}))^{-1} \exp\left(-c(\rho(x, y)/t^{\frac{1}{m}})^{\frac{m}{m-1}}\right) \quad (x, y \in \Omega, t > 0).$$

This is satisfied in particular by sublaplacian operators on Lie groups of polynomial growth [60] as considered e.g. in [47, 11, 1, 50, 20], or by more general elliptic and sub-elliptic operators [15, 52], and Schrödinger operators [53]. It is also satisfied by all the operators in [21, Section 2].

Examples of operators satisfying a generalized Gaussian estimate for  $p_0 > 1$  are higher order operators with bounded coefficients and Dirichlet boundary conditions on domains of  $\mathbb{R}^d$ , Schrödinger operators with singular potentials on  $\mathbb{R}^d$  and elliptic operators on Riemannian manifolds as listed in [3, Section 2] and the references therein.

**Theorem 8.3.** Assumption 8.2 implies that for any  $p \in (p_0, p'_0)$ ,

$$\left\{ \left( \frac{|z|}{\operatorname{Re} z} \right)^\alpha \exp(-zA) : \operatorname{Re} z > 0 \right\} \text{ is } R\text{-bounded on } L^p(\Omega),$$

where  $\alpha = d \left| \frac{1}{p_0} - \frac{1}{2} \right|$ . Consequently, by Theorem 7.1,  $A$  has an  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus on  $L^p(\Omega)$  for any  $\beta > d \left| \frac{1}{p_0} - \frac{1}{2} \right| + \frac{1}{2}$ .

*Proof.* By [5, Proposition 2.1], the assumption (GGE) implies that

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_0 \rightarrow 2} \leq C_1 \mu(B(x,r_t))^{\frac{1}{2} - \frac{1}{p_0}} \exp(-c_1(\rho(x,y)/r_t)^{\frac{m}{m-1}}) \quad (x, y \in \Omega, t > 0)$$

for some  $C_1, c_1 > 0$ . By [4, Theorem 2.1], this can be extended from real  $t$  to complex  $z = te^{i\theta}$  with  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$  :

$$\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\|_{p_0 \rightarrow 2} \leq C_2 \mu(B(x,r_z))^{\frac{1}{2} - \frac{1}{p_0}} (\cos \theta)^{-d(\frac{1}{p_0} - \frac{1}{2})} \exp(-c_2(\rho(x,y)/r_z)^{\frac{m}{m-1}}),$$

for  $r_z = (\cos \theta)^{-\frac{m-1}{m}} t^{\frac{1}{m}}$ , and some  $C_2, c_2 > 0$ . By [5, Proposition 2.1 (i) (1)  $\Rightarrow$  (3)] with  $R = e^{-zA}$ ,  $\gamma = \alpha = \frac{1}{p_0} - \frac{1}{2}$ ,  $\beta = 0$ ,  $r = r_z$ ,  $u = p_0$  and  $v = 2$ ], this gives for any  $x \in \Omega$ ,  $\operatorname{Re} z > 0$  and  $k \in \mathbb{N}_0$

$$\|\chi_{B(x,r_z)} e^{-zA} \chi_{A(x,r_z,k)}\|_{p_0 \rightarrow 2} \leq C_3 \mu(B(x,r_z))^{\frac{1}{2} - \frac{1}{p_0}} (\cos \theta)^{-d(\frac{1}{p_0} - \frac{1}{2})} \exp(-c_3 k^{\frac{m}{m-1}}),$$

where  $A(x, r_z, k)$  denotes the annular set  $B(x, (k+1)r_z) \setminus B(x, kr_z)$ . By [41, Theorem 2.2 with  $q_0 = p_0$ ,  $q_1 = s = 2$ ,  $\rho(z) = r_z$  and  $S(z) = (\cos \theta)^{d(\frac{1}{p_0} - \frac{1}{2})} e^{-zA}$ ] and property  $(\alpha)$ , we deduce that

$$\{(\cos \theta)^{d(\frac{1}{p_0} - \frac{1}{2})} e^{-zA} : \operatorname{Re} z > 0\}$$

is  $R$ -bounded. The first part of the theorem is shown. Then the second part follows from Theorem 7.1, noting that  $A$  has an  $H^\infty$  calculus on  $L^p(\Omega)$  [4, Corollary 2.3].  $\square$

**Remark 8.4.** Theorem 8.3 improves (if  $p_0 > 1$ ) the smoothness order of the spectral multiplier theorem in [3, Theorem 1.1] with same assumptions, from  $\frac{d}{2} + \frac{1}{2} + \epsilon$  in

[3] to  $d \left| \frac{1}{p_0} - \frac{1}{2} \right| + \frac{1}{2} + \epsilon$ . Note that [3] obtains also a weak-type result for  $p = p_0$ .

In [42], [59, Theorem 6.4 a)], under the assumptions of Theorem 8.3, a  $\mathcal{H}_r^\gamma$  calculus with  $\gamma > (d+1)|\frac{1}{p} - \frac{1}{2}|$  and  $r > |\frac{1}{2} - \frac{1}{p}|^{-1}$  is derived. Note that  $\mathcal{H}_r^\gamma$  is larger than  $\mathcal{H}_2^\beta$  by the Sobolev estimate in Lemma 3.2. In the classical case of Gaussian estimates, i.e.  $p_0 = 1$ , [21] yields a  $\mathcal{H}_\infty^{\alpha_2}$  calculus under Assumption 8.2 and even a  $\mathcal{H}_2^{\alpha_2}$  calculus for many examples, e.g. homogeneous operators, with the better derivation order  $\alpha_2 > \frac{d}{2}$ . In [10], further improvements on the differentiation order are obtained by making additional assumptions on the operator  $A$ , e.g. “restriction estimates”. However, Theorem 8.3 improves on all the above cited spectral multiplier theorems in that it includes the  $R$ -boundedness of the Hörmander calculus.

**Remark 8.5.** The theorem also holds for the weaker assumption that  $\Omega$  is an open subset of a homogeneous space  $\tilde{\Omega}$ . In that case, the ball  $B(x, r_t)$  on the right hand side in (GGE) is the one in  $\tilde{\Omega}$ . This variant can be applied to elliptic operators on irregular domains  $\Omega \subset \mathbb{R}^d$  as discussed in [3, Section 2].

**Remark 8.6.** In Theorem 8.3, the operator  $A$  was assumed to be self-adjoint, and thus, admits a functional calculus  $L^\infty \rightarrow B(L^2(\Omega))$ . The space  $L^\infty = L^\infty((0, \infty); d\mu_A)$  is larger than  $\mathcal{H}_p^\alpha$ , and one can use this fact to ameliorate the functional calculus of  $A$  on  $L^q(\Omega)$  by complex interpolation. One obtains that under the hypotheses of Theorem 8.3,  $A$  has a  $\mathcal{H}_q^\alpha$  calculus on  $L^p(\Omega)$  with  $p \in (p_0, p'_0)$  and

$$\alpha > d \left( \frac{1}{p_0} - \frac{1}{2} \right) \frac{\left| \frac{1}{p} - \frac{1}{2} \right|}{\frac{1}{p_0} - \frac{1}{2}} + \frac{1}{q} \quad \text{and} \quad \frac{1}{q} > \frac{1}{2} \frac{\left| \frac{1}{p} - \frac{1}{2} \right|}{\frac{1}{p_0} - \frac{1}{2}}.$$

**Remark 8.7.** In a forthcoming paper [17], we will also show a Hörmander theorem for  $A \otimes \text{id}_Y$  on  $X = L^p(\Omega; Y)$  for many self-adjoint  $A$  such that  $\exp(-tA)$  has Gaussian estimates, where  $Y$  is any UMD Banach lattice.

In the rest of this section, we show by way of examples and counterexamples to what extent our main theorems from Sections 6 and 7 are optimal.

**8.3. In (7.1),(7.2),(7.5),(7.6),  $R$ -bounds cannot be replaced by simple norm bounds in Theorems 7.1 and 7.4.** We will show this by way of counterexamples.

**Theorem 8.8.** Let  $\alpha \in (0, 1)$ . Then there exists a 0-sectorial operator on a super-reflexive space  $X$  with property  $(\alpha)$  such that

- (1)  $A$  has a bounded  $H^\infty(\Sigma_\omega)$  calculus for any  $\omega > 0$ .
- (2)  $A$  does not have a bounded  $\mathcal{H}_2^\beta$  calculus for any  $\beta > \frac{1}{2}$ .
- (3)  $\left\{ \left( \frac{\text{Re } z}{z} \right)^\alpha \exp(-zA) : \text{Re } z > 0 \right\}$  is bounded.
- (4)  $\left\{ (1 + |s|A)^{-(\alpha+\epsilon)} e^{isA} : s \in \mathbb{R} \right\}$  is bounded for any  $\epsilon > 0$ .
- (5) The sets in (3) and (4) are not  $R$ -bounded even if  $\alpha$  is replaced by any large number  $\alpha'$ .

Moreover, there is the following variant. Let  $\theta \in (0, \pi)$ . There exists a 0-sectorial operator on a super-reflexive space  $X$  with property  $(\alpha)$  such that

- (1)  $A$  has a bounded  $H^\infty(\Sigma_\omega)$  calculus for any  $\omega > \alpha\theta$  but for no  $\omega < \alpha\theta$ .
- (2) Statements (2) - (5) above hold.

*Proof.* We first prove the variant. We take the operator  $A$  from [33, Theorem 2.4 and Proposition 2.5]. It is the multiplication operator  $Af(x) = e^x f(x)$  defined initially on  $L^2(\mathbb{R})$ . It is then first extended to the space  $\mathcal{H}_\theta$  defined as the completion of  $L^2(\mathbb{R})$  with respect to the norm  $\|f\|_\theta^2 = \int_{\mathbb{R}} e^{-2\theta|\xi|} |\hat{f}(\xi)|^2 d\xi$ , and second to the space  $X_\theta$  defined as the completion of  $L^2(\mathbb{R})$  with respect to the norm  $\|f\|_{X_\theta} = \sup_{a \in \mathbb{R}} \|f\chi_{(-\infty, a]}\|_\theta$ . Finally,  $A$  is regarded on the complex interpolation space  $X = [L^2, X_\theta]_\alpha$ . By [14],  $X$  is uniformly convex and thus super-reflexive. Further, at the end of this proof we will show that  $X$  has property  $(\alpha)$ . It is shown in [33, p. 98] that  $A$  is 0-sectorial with dense domain and dense range on  $X$ . Further, it is shown in [33, Proposition 2.5] that condition (1) of the proposition holds ( $A$  acting on  $X$ ).

Since  $H^\infty(\Sigma_\omega) \hookrightarrow \mathcal{H}_2^\beta$  for any  $\omega \in (0, \pi)$  and  $\beta > \frac{1}{2}$ , (1) implies that  $A$  cannot have a  $\mathcal{H}_2^\beta$  calculus, so (2) is shown.

This also implies (5) since the  $R$ -boundedness of one of the sets in (3) and (4) would imply a  $\mathcal{H}_2^\beta$  calculus according to Theorems 7.1 and 7.4.

It remains to check (3) and (4). For (3), we remark that  $\exp(-e^{i\sigma} r A)f(x) = e_{e^{i\sigma} r}(x)f(x)$  for  $f \in L^2(\mathbb{R})$  and  $e_{e^{i\sigma} r}(x) = \exp(-e^{i\sigma} r e^x)$ . Similarly to [33, Proof of Theorem 2.4], we have

$$e_{e^{i\sigma} r} f = \int_{\mathbb{R}} -e'_{e^{i\sigma} r}(x) f \chi_{(-\infty, x)} dx$$

as a Bochner integral in  $L^2(\mathbb{R})$ , and consequently,

$$\|e_{e^{i\sigma} r} f\|_{X_\theta} \leq \int_{\mathbb{R}} |e'_{e^{i\sigma} r}(x)| dx \|f\|_{X_\theta}.$$

We have

$$\int_{\mathbb{R}} |e'_{e^{i\sigma} r}(x)| dx = \int_{\mathbb{R}} r e^x |\exp(-r e^{i\sigma} e^x)| dx$$

$$\begin{aligned}
&= \int_0^\infty r \exp(-r \cos(\sigma)u) du \\
&= \int_0^\infty \exp(-u) \frac{du}{\cos(\sigma)} \\
&= \frac{1}{\cos(\sigma)}.
\end{aligned}$$

It follows that on  $X_\theta$ , we have  $\|\exp(-zA)\|_{X_\theta \rightarrow X_\theta} \leq \frac{|z|}{\operatorname{Re} z}$ . Moreover, on  $L^2(\mathbb{R})$ , we have  $\|\exp(-re^{i\sigma}A)\|_{L^2 \rightarrow L^2} = \sup_{x \in \mathbb{R}} |\exp(-re^{i\sigma}e^x)| = 1$ . By complex interpolation, it follows that

$$\|\exp(-zA)\|_{X \rightarrow X} \leq \|\exp(-zA)\|_{L^2 \rightarrow L^2}^{1-\alpha} \|\exp(-zA)\|_{X_\theta \rightarrow X_\theta}^\alpha \leq \left( \frac{|z|}{\operatorname{Re} z} \right)^\alpha.$$

Thus, (3) is shown.

For (4), we argue similarly and replace  $e_{e^{i\sigma}r}$  by  $f_{s,\beta}(x) = (1 + |s|e^x)^{-\beta} \exp(ise^x)$ . Then

$$\begin{aligned}
\|f_{s,\beta}(A)\|_{X_\theta \rightarrow X_\theta} &\lesssim \int_{\mathbb{R}} |f'_{s,\beta}(x)| dx = \int_0^\infty \left| \frac{d}{du} [f_{s,\beta}(\log(u))] \right| du \\
&\lesssim \int_0^\infty (1 + |s|u)^{-(\beta+1)} |s| + |s|(1 + |s|u)^{-\beta} du \\
&= \int_0^\infty (1 + u)^{-(\beta+1)} + (1 + u)^{-\beta} du \\
&< \infty
\end{aligned}$$

as soon as  $\beta > 1$ . To get an estimate on  $X = [L^2, X_\theta]_\alpha$ , apply complex interpolation to the analytic family of operators  $\beta \mapsto f_{s,\beta}(A)$ , to deduce

$$\|f_{s,\beta}(A)\|_{X \rightarrow X} \leq \|f_{s,0}(A)\|_{L^2 \rightarrow L^2}^{1-\alpha} \|f_{s,1+\epsilon}(A)\|_{X_\theta \rightarrow X_\theta}^\alpha < \infty$$

for  $\beta = \alpha(1 + \epsilon)$ , i.e.  $\beta > \alpha$ .

Now we prove the first part of Theorem 8.8, i.e. the  $H^\infty$  calculus angle will be arbitrarily small. Indeed, we can modify Kalton's example from [33] in the following way. Let  $\mathcal{H}_w$  be the completion of  $L^2(\mathbb{R})$  with respect to the norm  $\|f\|_w^2 = \int_{\mathbb{R}} w(\xi)^2 |\hat{f}(\xi)|^2 d\xi$ , where  $w : \mathbb{R} \rightarrow (0, 1]$  is a weight function. Hence the original example in [33] considers  $\mathcal{H}_w = \mathcal{H}_\theta$  with  $w(\xi) = \exp(-\theta|\xi|)$ . Further take  $X_w$  the completion of  $L^2(\mathbb{R})$  with respect to the norm  $\|f\|_{X_w} = \sup_{a \in \mathbb{R}} \|f\chi_{(-\infty, a]}\|_w$ , and finally  $X = [L^2, X_w]_\alpha$ , the complex interpolation space. Again, by [14],  $X$  is uniformly convex, thus super-reflexive, and we will show at the end that  $X$  has property  $(\alpha)$ . The operator  $A$  is again the multiplication operator  $Af(x) = e^x f(x)$ . Then as in the first part of the proof, one can show that (3) and (4) hold.

Note that

$$\|A^{it}\|_{X \rightarrow X} \cong \|A^{it}\|_{\mathcal{H}_w \rightarrow \mathcal{H}_w} \cong \sup_{\xi \in \mathbb{R}} \frac{w(\xi + t)^\alpha}{w(\xi)^\alpha},$$

where the first equivalence can be shown as in [33, Proof of Proposition 2.5] (the new weight  $w^\alpha$  is the interpolated weight of  $w_0(\xi) = 1$  for  $L^2$  and  $w_1(\xi) = w(\xi)$ ), and the second follows easily from the equality  $(A^{it}f)^\wedge(\xi) = \hat{f}(\xi - t)$ .

Choose now the weight  $w$  such that  $\sup_{\xi \in \mathbb{R}} \frac{w(\xi+t)^\alpha}{w(\xi)^\alpha}$  grows subexponentially in  $t$ , but superpolynomially in  $t$ , e.g.  $w(\xi) = \exp(-\sqrt{|\xi|})$  works. Then  $A$  has a bounded  $H^\infty$  calculus on  $X$  since it has one on  $L^2$  and  $\mathcal{H}_w$ , and the arguments in [33, Proofs

of Theorem 2.4 and Proposition 2.5] apply literally. Then on the one hand the subexponential growth  $\|A^{it}\|_{X \rightarrow X} \leq C_\epsilon e^{|t|^\epsilon}$  implies that  $A$  has an  $H^\infty(\Sigma_\omega)$  calculus for any angle  $\omega > 0$ , and the superpolynomial growth  $\|A^{it}\|_{X \rightarrow X} \geq c_\beta (1 + |t|)^\beta$  for any  $\beta > 0$  implies by Lemmas 3.9 and 3.2 that  $A$  cannot have a  $\mathcal{H}_2^\gamma$  calculus for any  $\gamma > 0$ .

Let us now show that  $X = [L^2, X_w]_\alpha$  has property  $(\alpha)$  for  $w(\xi)$  either  $e^{-\theta|\xi|}$  or  $\exp(-\sqrt{|\xi|})$  (or any other bounded weight), thus finishing the proof of both parts above. Since  $X$  is super-reflexive, it has finite cotype. Thus, according to [54], it suffices to show that  $X$  is a subspace of a space with local unconditional structure. We will show that

$$(8.2) \quad X = [L^2, X_w]_\alpha \hookrightarrow L^\infty(\mathbb{R}, [L^2, \mathcal{H}_w]_\alpha),$$

more precisely,  $X$  is isomorphic to a subspace of the right hand side. Considering functions in  $L^2$  and  $\mathcal{H}_w$  in their Fourier image, these spaces are weighted  $L^2$ -spaces, hence their complex interpolation is a weighted  $L^2$  space again [2, 5.5.3 Theorem], and in particular a Banach lattice. Then  $L^\infty(\mathbb{R}, [L^2, \mathcal{H}_w]_\alpha)$  is a Banach lattice and hence a subspace of a space with local unconditional structure. It therefore only remains to show (8.2). To this end, consider

$$j_0 : L^2 \rightarrow L^\infty(\mathbb{R}, L^2), f \mapsto (a \mapsto f\chi_{(-\infty, a]})$$

and

$$j_1 : X_w \rightarrow L^\infty(\mathbb{R}, \mathcal{H}_w), f \mapsto (a \mapsto f\chi_{(-\infty, a]}).$$

By definition of  $X_w$ ,  $j_1$  is an isometric embedding, and it is easy to see that  $j_0$  also is an isometric embedding. Both  $j_0$  and  $j_1$  have a left inverse. Indeed, for  $g \in L^2$ , let  $P_g$  be the linear form

$$P_g : L^\infty(\mathbb{R}, L^2) \rightarrow \mathbb{C}, (a \mapsto f_a) \mapsto \lim_{a \rightarrow \infty} \langle f_a, g \rangle,$$

where  $\lim_{a \rightarrow \infty}$  is any Banach limit (along integers  $a \in \mathbb{N}$ , say). It is easy to check that the linear form  $P(f_a) : L^2 \rightarrow \mathbb{C}, g \mapsto P_g(f_a)$  is bounded by  $\|f_a\|_{L^\infty(\mathbb{R}, L^2)} \|g\|_{L^2}$ , so that by the Riesz representation theorem, we have constructed a contraction  $P : L^\infty(\mathbb{R}, L^2) \rightarrow L^2, (f_a) \mapsto P(f_a)$ . Further, it is easy to check that  $P \circ j_0$  is the identity on  $L^2$ . For  $j_1$ , consider the analogous construction

$$Q_g : L^\infty(\mathbb{R}, \mathcal{H}_w) \rightarrow \mathbb{C}, (a \mapsto f_a) \mapsto \lim_{a \rightarrow \infty} \langle f_a, g \rangle_{\mathcal{H}_w},$$

and set  $Q(f_a) : g \mapsto Q_g(f_a)$ . One has  $\langle Qj_1(f), g \rangle_{\mathcal{H}_w} = \lim_{a \rightarrow \infty} \langle f\chi_{(-\infty, a]}, g \rangle_{\mathcal{H}_w}$ , and it is easy to check by Plancherel's theorem that for  $f, g \in L^2 \subseteq \mathcal{H}_w$ , this equals  $\langle f, g \rangle_{\mathcal{H}_w}$ . Now conclude by density of  $L^2$  in  $\mathcal{H}_w$  that  $Qj_1$  is the identity on  $\mathcal{H}_w$ . Then by the retraction theorem [58, Section 1.2.4], we have that

$$X = [L^2, X_w]_\alpha \text{ is a complemented subspace of } [L^\infty(\mathbb{R}, L^2), L^\infty(\mathbb{R}, \mathcal{H}_w)]_\alpha.$$

Note that  $j_0$  resp.  $j_1$  take actually values in  $L_0^\infty(\mathbb{R}, L^2)$  resp.  $L_0^\infty(\mathbb{R}, \mathcal{H}_w)$ , the closure of those functions that can be written as a step function  $v(a) = \sum_{k=1}^N \chi_{A_k}(a) f_k$ , with  $A_k \subseteq \mathbb{R}$  measurable and  $f_k \in L^2$  resp.  $\in \mathcal{H}_w$ . Indeed, using dominated convergence, it is easy to check that

$$v(a) = \chi_{[\frac{1}{N}(N+1), \infty)}(a) f + \sum_{k=-N}^N \chi_{[\frac{1}{N}k, \frac{1}{N}(k+1))}(a) f \chi_{(-\infty, \frac{1}{N}k]}$$

approximates  $j_0(f)$  in  $L^\infty(\mathbb{R}, L^2)$  as  $n, N \rightarrow \infty$ . Thus,  $j_0(L^2) \subseteq L_0^\infty(\mathbb{R}, L^2)$ . Since  $L^2 \hookrightarrow \mathcal{H}_w$ , one also has  $j_1(L^2) \subseteq L_0^\infty(\mathbb{R}, \mathcal{H}_w)$ , so by density of  $L^2$  in  $\mathcal{H}_w$ ,  $j_1(\mathcal{H}_w) \subseteq L_0^\infty(\mathbb{R}, \mathcal{H}_w)$ . We infer that  $X$  is a complemented subspace of  $[L_0^\infty(\mathbb{R}, L^2), L_0^\infty(\mathbb{R}, \mathcal{H}_w)]_\alpha$ . We consider the  $L_0^\infty$  spaces on  $\mathbb{R}$  with finite measure, say,  $\frac{1}{1+a^2}da$ , noting that any change of measure results in an isometry on the  $L^\infty$  level. Then the simple functions (step functions such that the  $A_k$  have finite measure) with values in  $L^2$  are dense in  $L_0^\infty(\mathbb{R}, L^2) \cap L_0^\infty(\mathbb{R}, \mathcal{H}_w) = L_0^\infty(\mathbb{R}, L^2)$ . We infer that the proof of [2, 5.1.2 Theorem] applies, showing that the natural map  $[L_0^\infty(\mathbb{R}, L^2), L_0^\infty(\mathbb{R}, \mathcal{H}_w)]_\alpha \hookrightarrow L^\infty(\mathbb{R}, [L^2, \mathcal{H}_w]_\alpha)$  is an isomorphic embedding onto a closed subspace. We have shown (8.2) and the proof of the theorem is complete.  $\square$

#### 8.4. The bounded $H^\infty$ calculus assumption in Theorems 6.1 and 7.1 is necessary.

a) The bounded  $H^\infty$  calculus in the assumption of Theorem 6.1 cannot be omitted as the following example of a 0-sectorial operator with uniformly bounded imaginary powers, but without any bounded  $H^\infty$  calculus shows. By Lemma 3.2, we have  $H^\infty(\Sigma_\sigma) \hookrightarrow \mathcal{H}_r^\beta$  for any  $\sigma \in (0, \pi)$ ,  $r \in [1, \infty)$  and  $\beta > \frac{1}{r}$  so that this operator cannot have a  $\mathcal{H}_r^\beta$  calculus.

Consider the 0-sectorial operator  $A$  on  $L^p(\mathbb{R})$  such that  $A^{it}$  is the shift group, i.e.  $A^{it}g = g(\cdot + t)$ . By [13, Lemma 5.3],  $A$  does not have an  $H^\infty(\Sigma_\sigma)$  calculus to any positive  $\sigma$ , unless  $p = 2$ . However,  $A^{it}$  is even uniformly bounded, so that the assumptions of Theorem 6.1 (2) hold for  $\alpha = 0$ , and also for (1) according to Remark 2.4.

b) The bounded  $H^\infty$  calculus in the assumption of Theorem 7.1 cannot be omitted. We give an example below of a 0-sectorial operator without an  $H^\infty(\Sigma_\sigma)$  calculus for any  $\sigma \in (0, \pi)$ , but satisfying the assumptions of Theorem 7.1 part (1) or (2) with  $\alpha = 1$ , on a Hilbert space. By Lemma 3.2, we have  $H^\infty(\Sigma_\sigma) \hookrightarrow \mathcal{H}_r^\beta$  for any  $\sigma \in (0, \pi)$ ,  $r \in [1, \infty)$  and  $\beta > \frac{1}{r}$ , so that this operator cannot have a  $\mathcal{H}_r^\beta$  calculus.

Our example will satisfy

$$(8.3) \quad \|T(e^{i\theta}t)\| \lesssim \left(\frac{\pi}{2} - |\theta|\right)^{-1} \quad (t > 0, |\theta| < \frac{\pi}{2}).$$

In [44, Theorem 4.1], the following situation is considered, based on an idea of Baillon and Clément. Let  $X$  be an infinite dimensional space admitting a Schauder basis  $(e_n)_{n \geq 1}$ . Let  $V$  denote the span of the  $e_n$ 's. For a sequence  $a = (a_n)_{n \geq 1}$ , the operator  $T_a : V \rightarrow V$  is defined by letting  $T_a(\sum_n \alpha_n e_n) = \sum_n a_n \alpha_n e_n$  for any finite family  $(\alpha_n)_{n \geq 1} \subset \mathbb{C}$ . Let  $a^{(N)} = (a_n^{(N)})_{n \geq 1}$  be the sequence defined by  $a_n^{(N)} = \delta_{n \leq N}$ . It is well-known that for any Schauder basis (even conditional),  $T_{a^{(N)}}$  extends to a bounded projection on  $X$  and  $\sup_N \|T_{a^{(N)}}\| < \infty$  [46, Chapter 1]. This readily implies that for any sequence  $a = (a_n)_{n \geq 1}$  of bounded variation,  $T_a$  extends to a bounded operator, and

$$(8.4) \quad \|T_a\| \lesssim \|a\|_{BV} := \|a\|_{\ell^\infty} + \sum_{n=1}^{\infty} |a_n - a_{n+1}|.$$

In [44], it is shown that for  $a_n = 2^{-n}$ , the bounded linear extension  $A : X \rightarrow X$  of  $T_a$  is a 0-sectorial (injective) operator, and that for  $f \in H^\infty(\Sigma_\sigma)$ ,  $V$  is a subset of

$D(f(A))$ . Further, for  $x \in V$ , one has

$$(8.5) \quad f(A)x = T_{f(a)}x,$$

where  $f(a)_n = f(a_n)$ . Finally, it is shown in [44] that if the Schauder basis is conditional, then  $A$  does not have a bounded  $H^\infty$  calculus.

Now assume that  $X$  is a separable Hilbert space, so that the assumptions of Theorem 7.1 (1) and (2) reduce to norm boundedness in (8.3). Clearly,  $X$  admits a Schauder basis, and as mentioned in [44] even a conditional one. We take a conditional basis and consider the operator  $A$  above without a bounded  $H^\infty$  calculus. By (8.4) and (8.5), (8.3) will follow from

$$(8.6) \quad \|(\exp(-te^{i\theta}2^{-n}))_n\|_{BV} \lesssim \left(\frac{\pi}{2} - |\theta|\right)^{-1} \quad (t > 0, |\theta| < \frac{\pi}{2}).$$

It is easy to check that

$$|\exp(-te^{i\theta}2^{-n}) - \exp(-te^{i\theta}2^{-(n+1)})| \lesssim 2^{-(n+1)}t \exp(-t \cos(\theta)2^{-(n+1)}).$$

Thus,

$$\begin{aligned} \|(\exp(-te^{i\theta}2^{-n}))_n\|_{BV} &\lesssim 1 + \sum_{n=1}^{\infty} 2^{-(n+1)}t \exp(-t \cos(\theta)2^{-(n+1)}) \\ &\lesssim 1 + \int_0^1 st \exp(-t \cos(\theta)s) \frac{ds}{s} \\ &\lesssim 1 + (\cos \theta)^{-1} \int_0^\infty \exp(-s) ds \\ &\cong \left(\frac{\pi}{2} - |\theta|\right)^{-1}. \end{aligned}$$

This shows (8.6), and thus  $A$  satisfies (8.3) without having an  $H^\infty$  calculus.

**8.5. The order of the resulting  $\mathcal{H}_2^\beta$  calculus in Theorems 6.1, 7.1 and 7.4 cannot be improved in general.** In Theorems 6.1, 7.1 and 7.4, we assume one of several conditions on the differentiation parameter  $\alpha$  and conclude the boundedness for a  $\mathcal{H}_p^\beta$  calculus with some  $\beta > \alpha$ . We give now a list of examples showing that the loss from  $\alpha$  to  $\beta$  is close to being optimal.

*a) The gap between assumptions on the parameter  $\alpha$  in (1) and (2) of Theorems 6.1, 7.1 and 7.4, and the order  $\alpha + \frac{1}{2}$  of the  $R$ -bounded  $\mathcal{H}_2^{\alpha+\frac{1}{2}}$  calculus is optimal, even in a Hilbert space.*

*Proof.* Let  $\alpha = m \in \mathbb{N}$  be given and consider the Jordan block

$$B = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{C}^{(m+1) \times (m+1)}$$

as an operator on the Hilbert space  $X = \ell_{m+1}^2$  and set  $A = e^B$ . Then

$$f \circ \log(A) = \begin{pmatrix} f(0) & \frac{f'(0)}{1!} & \cdots & \frac{f^{(m)}(0)}{m!} \\ 0 & \ddots & \ddots & \end{pmatrix},$$



at least for  $f \circ \log \in \bigcup_{\omega \in (0, \pi)} \text{Hol}(\Sigma_\omega)$ . Thus, for  $x = (x_0, \dots, x_m)$  and  $y = (y_0, \dots, y_m) \in \ell_{m+1}^2$ ,

$$|\langle A^{it}x, y \rangle| = \left| \sum_{k=0}^m \sum_{l=k}^m \frac{(it)^{l-k}}{(l-k)!} x_l y_k \right| \lesssim \langle t \rangle^m \|x\| \|y\|,$$

and taking  $x = (0, \dots, 0, 1)$ ,  $y = (1, 0, \dots, 0)$  shows that the exponent  $m$  is optimal in this estimate. Thus,  $t \mapsto \langle t \rangle^{-\beta} \langle A^{it}x, y \rangle$  belongs to  $L^2(\mathbb{R})$  for all  $x, y \in X$  if and only if  $\beta > m + \frac{1}{2}$ , so that  $A$  cannot have a  $\mathcal{H}_2^\beta$  calculus for  $\beta \leq m + \frac{1}{2}$ . We refer to [39] for an adequate adaptation of the  $R$ -boundedness notion which is equivalent to the  $R$ -bounded  $\mathcal{H}_2^\beta$  calculus.

On the other hand,  $\|A^{it}\| \cong \langle t \rangle^m$ , so that the assumption in (2) of Theorem 6.1 holds with  $\alpha = m$ , and since  $X$  is a Hilbert space, also the assumption in (1). Furthermore the assumptions in (1) and (2) of Theorem 7.4 and thus, by Proposition 7.5, also of Theorem 7.1 hold, because

$$\|(1 + |t|A)^{-m} e^{itA}\| \cong \left| \frac{d^m}{ds^m} \left[ (1 + |t|e^s)^{-m} \exp(ite^s) \right]_{s=0} \right| \leq C.$$

Therefore, in this example, assumptions (1) and (2) of Theorems 6.1, 7.1 and 7.4 hold with  $\alpha = m$ , but  $A$  does not have an  $R$ -bounded  $\mathcal{H}_2^{m+\frac{1}{2}}$  calculus. This shows that part (1) of Theorems 6.1, 7.1 and 7.4 is sharp.  $\square$

*b) The difference of  $\alpha$  in the  $R$ -bounded  $\mathcal{H}_r^\alpha$  calculus and assumption (2) in Theorem 6.1.*

*Proof.* The  $R$ -bounded  $\mathcal{H}_r^\alpha$  calculus implies essentially any of the assumptions in (1) and (2) of Theorems 6.1, 7.1 and 7.4 with the same parameter  $\alpha$ , but in the converse direction, we have to increase the order of the Hörmander calculus by essentially  $\max(\frac{1}{2}, \frac{1}{\text{type } X} - \frac{1}{\text{cotype } X})$ . An example showing the optimality of this gap in the case of Theorem 6.1 is again a multiplication operator, on a different space, it can be found in [35, Theorem 5.26].  $\square$

## 9. BOCHNER-RIESZ MEANS AND THE $\mathcal{H}_1^\alpha$ CALCULUS

For  $\alpha > 1$  and  $u > 0$ , we let  $R_u^{\alpha-1}(\lambda) = (1 - \lambda/u)_+^{\alpha-1}$  be the Bochner-Riesz functions, where  $t_+ = \max(t, 0)$ . In the next theorem, we show that under some mild assumptions that in the presence of an  $H^\infty$  calculus,  $R$ -boundedness of Bochner-Riesz means and an  $R$ -bounded  $\mathcal{H}_1^\alpha$  calculus are essentially equivalent.

**Theorem 9.1.** Let  $1 < \beta < \alpha$  and assume that  $A$  is 0-sectorial and has an auxiliary calculus  $\Phi_A : \mathcal{H}_1^\beta \rightarrow B(D(\theta), X)$  for some  $\theta \geq 0$ , and an  $H^\infty(\Sigma_\omega)$  functional calculus for some  $\omega \in (0, \pi/2)$ .

- (1) If  $A$  has an  $R$ -bounded  $\mathcal{H}_1^\beta$  functional calculus, then  $\{R_u^{\alpha-1}(A) : u > 0\}$  is  $R$ -bounded in  $X$ .
- (2) If  $\{R_u^{\alpha-1}(A) : u > 0\}$  is  $R$ -bounded in a Banach space  $X$  (resp. in a space with property  $(\alpha)$ ), then  $A$  has a bounded (resp.  $R$ -bounded)  $\mathcal{H}_1^\alpha$  functional calculus.

*Proof.* (1) It clearly suffices to note that  $\sup_{u>0} \|R_u^{\alpha-1}\|_{\mathcal{H}_1^\beta}$  is finite, which has been proven in Lemma 3.9 (4).

(2) Note that the auxiliary calculus plus the  $H^\infty(\Sigma_\omega)$  calculus allows to define  $R_u^{\alpha-1}(A)$  as closed operators with dense domain. We want to apply Theorem 5.1 and check condition (5.1). So let  $f \in C_c^\infty(\mathbb{R}_+)$  with  $\text{supp } f \subset [\frac{1}{2}, 2]$  and  $\|f\|_{W_1^\alpha} \leq 1$ . Let  $f^{(\alpha)}$  be the Cossar-Riemann-Liouville derivative of  $f$ , as defined e.g. in [26, p. 1011]. For functions with compact support, one has  $f^{(\alpha)}(\xi) = (-i\xi)^\alpha \hat{f}(\xi)$ ,  $\xi \in \mathbb{R}$ , so that in particular  $\|f^{(\alpha)}\|_{L^1} \lesssim \|f\|_{W_1^\alpha}$ . This also implies that  $(f(a\cdot))^{(\alpha)}(t) = a^\alpha f^{(\alpha)}(at)$ . By Lemma 4.6 (4), for  $x \in D_A$ ,

$$f(2^n A)x = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty 2^{n\alpha} f^{(\alpha)}(2^n u)(u-A)_+^{\alpha-1} x du = \frac{(-1)^m}{\Gamma(\alpha)} \int_0^\infty 2^{n\alpha} f^{(\alpha)}(2^n u) u^{\alpha-1} R_u^{\alpha-1}(A)x du,$$

where  $m = \lfloor \alpha \rfloor$ . According to the assumptions,  $\{R_u^{\alpha-1}(A) : u > 0\}$  is  $R$ -bounded, so that  $\{f(2^n A) : n \in \mathbb{Z}, f \text{ as above}\}$  is  $R$ -bounded if  $\|2^{n\alpha} f^{(\alpha)}(2^n u) u^{\alpha-1}\|_{L^1(du)}$  can be bounded independently of  $n \in \mathbb{Z}$  and  $f$  as above. But we have, since  $\alpha > 1$ ,

$$\begin{aligned} \int_0^\infty |2^{n\alpha} f^{(\alpha)}(2^n u) u^{\alpha-1}| du &= \int_0^\infty |f^{(\alpha)}(u) u^{\alpha-1}| du = \int_0^2 |f^{(\alpha)}(u)| u^{\alpha-1} du \lesssim \int_0^2 |f^{(\alpha)}(u)| du \\ &\lesssim \|f\|_{W_1^\alpha} \leq 1. \end{aligned}$$

□

Note that the auxiliary calculus in the preceding Theorem was only assumed to be able to define the Bochner-Riesz means as closed densely-defined operators, it can be omitted if  $A$  is self-adjoint on  $L^2(U)$  and  $X = L^p(U)$ . To our knowledge,  $R$ -bounded Bochner-Riesz means have been considered for the first time in [7] and [56]. In [56, Theorem A], see also [7, Théorème (7.2)], a functional calculus similar to our  $\mathcal{H}_1^\beta$  calculus is established, with the somewhat stronger norm,

$$\|f\|_{L^\infty(\mathbb{R}_+)} + \max_{j=1,2,\dots,\beta} \sup_{R>0} \frac{1}{R} \int_0^R \lambda^j |f^{(j)}(\lambda)| d\lambda,$$

in place of  $\max_{j=0,1,\dots,\beta} \sup_{R>0} \frac{1}{R} \int_{R/2}^R \lambda^j |f^{(j)}(\lambda)| d\lambda$ . It is assumed there that the Bochner-Riesz means of a self-adjoint operator  $A$  are  $R$ -bounded on  $L^p$ ,  $1 < p < \infty$  and the semigroup generated by  $-A$  is contractive on all  $L^p$ ,  $1 \leq p \leq \infty$ . Note that a self-adjoint operator generating such a semigroup always has an  $H^\infty$  calculus on  $L^p$ ,  $1 < p < \infty$  [12, Theorem 2].

In the following proposition, we give a sufficient criterion for the  $R$ -boundedness of Bochner-Riesz means.

**Proposition 9.2.** Let  $A$  be a 0-sectorial operator and  $\nu > \alpha \geq 0$ . Assume that

$$\left\{ \left( \frac{\text{Re } z}{|z|} \right)^\alpha e^{-zA} : \text{Re } z > 0 \right\} \text{ is } R\text{-bounded.}$$

Then  $\{R_u^\nu(A) : u > 0\}$  is  $R$ -bounded, too.

*Proof.* According to [27, (4.2) and p. 332 Remark], see also [27, Proof of Lemma 6.1], under the hypotheses of the proposition, one has the formula

$$R_u^\nu(A) = u^{-\nu} \frac{1}{2\pi i} \int_{\text{Re } z = \frac{1}{u}} \frac{e^{-zA}}{z^{\nu+1}} e^{uz} dz.$$

If we write  $z = \frac{1}{u} + is$ , then  $|e^{uz}| = |e^{1+isu}| = e$ ,  $|z^{\nu+1}| = u^{-\nu-1}|1 + ius|^{\nu+1}$  and  $u^\alpha|z|^\alpha = |1 + ius|^\alpha$ . Hence

$$(9.1) \quad R_u^\nu(A) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \frac{1}{u}} \left[ e^{-zA} \left( \frac{\operatorname{Re} z}{|z|} \right)^\alpha \right] g_u(z) dz$$

with  $g_u(z) = u^{-\nu} \frac{e^{uz}}{z^{\nu+1}} \left( \frac{|z|}{\operatorname{Re} z} \right)^\alpha$ . We have  $|g_u(z)| = e|1 + ius|^{-\nu-1}|1 + ius|^\alpha u$ . Since  $\nu > \alpha$ ,  $\int_{\operatorname{Re} z = \frac{1}{u}} |g_u(z)| |dz| \leq e \int_{\mathbb{R}} |1 + ius|^{-(\nu-\alpha+1)} u ds \leq C$  with  $C$  independent of  $u$ . Now the proposition follows from (9.1) together with the assumptions and [43, Corollary 2.14].  $\square$

## 10. BISECTORIAL AND STRIP-TYPE OPERATORS

**10.1. Bisectorial operators.** In this short subsection we indicate how to extend our results to bisectorial operators. An operator  $A$  with dense domain on a Banach space  $X$  is called bisectorial of angle  $\omega \in [0, \frac{\pi}{2})$  if it is closed, its spectrum is contained in the closure of  $S_\omega = \{z \in \mathbb{C} : |\arg(\pm z)| < \omega\}$ , and one has the resolvent estimate

$$\|(I + \lambda A)^{-1}\|_{B(X)} \leq C_{\omega'}, \quad \forall \lambda \notin S_{\omega'}, \quad \omega' > \omega.$$

If  $X$  is reflexive, then for such an operator we have again a decomposition  $X = N(A) \oplus \overline{R(A)}$ , so that we may assume that  $A$  is injective. The  $H^\infty(S_\omega)$  calculus is defined as in (3.2), but now we integrate over the boundary of the double sector  $S_\omega$ . If  $A$  has a bounded  $H^\infty(S_\omega)$  calculus, or more generally, if we have  $\|Ax\| \cong \|(-A^2)^{\frac{1}{2}}x\|$  for  $x \in D(A) = D((-A^2)^{\frac{1}{2}})$  (see e.g. [19]), then the spectral projections  $P_1, P_2$  with respect to  $\Sigma_1 = S_\omega \cap \mathbb{C}_+$ ,  $\Sigma_2 = S_\omega \cap \mathbb{C}_-$  give a decomposition  $X = X_1 \oplus X_2$  of  $X$  into invariant subspaces for the resolvents of  $A$  such that the restriction  $A_1$  of  $A$  to  $X_1$  and  $-A_2$  of  $-A$  to  $X_2$  are sectorial operators with  $\sigma(A_i) \subset \Sigma_i$ . For  $f \in H_0^\infty(S_\omega)$  we have

$$(10.1) \quad f(A)x = f|_{\Sigma_1}(A_1)P_1x + f|_{\Sigma_2}(A_2)P_2x.$$

We define the Hörmander class  $\mathcal{H}_p^\alpha(\mathbb{R})$  on  $\mathbb{R}$  by restrictions:  $f \in \mathcal{H}_p^\alpha(\mathbb{R})$  iff  $f\chi_{\mathbb{R}_+} \in \mathcal{H}_p^\alpha$  and  $f(-\cdot)\chi_{\mathbb{R}_+} \in \mathcal{H}_p^\alpha$ . Let  $A$  be a 0-bisectorial operator, i.e.  $A$  is  $\omega$ -bisectorial for all  $\omega > 0$ . Then  $A$  has a  $\mathcal{H}_p^\alpha(\mathbb{R})$  calculus if there is a constant  $C$  so that  $\|f(A)\| \leq C\|f\|_{\mathcal{H}_p^\alpha(\mathbb{R})}$  for  $f \in \bigcap_{0 < \omega < \pi} H^\infty(S_\omega) \cap \mathcal{H}_p^\alpha(\mathbb{R})$ . Clearly,  $A$  has a  $\mathcal{H}_p^\alpha(\mathbb{R})$  calculus if and only if  $A_1$  and  $-A_2$  have a  $\mathcal{H}_p^\alpha$  calculus and in this case (10.1) holds again.

Let  $f_t(\lambda) = \begin{cases} \lambda^{it} : \operatorname{Re} \lambda > 0 \\ (-\lambda)^{it} : \operatorname{Re} \lambda < 0 \end{cases}$ . Then  $f_t \in H^\infty(S_\omega)$  for any  $\omega \in (0, \frac{\pi}{2})$ . Clearly,

one has  $f_t(A) = A_1^{it} \oplus (-A_2)^{it}$  on  $X = X_1 \oplus X_2$ . Similarly, let  $R_u^\alpha(\lambda) = (1 - |\lambda|/u)_+^\alpha$ , so that  $R_u^\alpha(A) = (1 - A_1/u)_+^\alpha \oplus (1 - (-A_2)/u)_+^\alpha$ . Finally, let  $g_s^\alpha(\lambda) = (i + |s|\lambda)^{-\alpha} e^{is\lambda}$ , so that  $g_s^\alpha(A) = (i + |s|A_1)^{-\alpha} e^{isA_1} \oplus (i - |s|(-A_2))^{-\alpha} e^{-i|s|(-A_2)}$ . Then by an obvious modification of the proof of Proposition 7.5, one can show that if  $\{g_s^\alpha(A) : s \in \mathbb{R}\}$  is  $R$ -bounded then (7.1) holds for both  $A_1$  and  $-A_2$ , and that  $\sup_{s \in \mathbb{R}} R(\{g_{2^k s}^\alpha(A) : k \in \mathbb{Z}\}) < \infty$  implies that (7.2) holds for both  $A_1$  and  $-A_2$ .

Then using the projections  $P_1$  and  $P_2$ , it is clear how our results from Sections 6, 7 and 9 extend to bisectorial operators.

**10.2. Operators of strip-type.** For  $\omega > 0$  we let  $\text{Str}_\omega = \{z \in \mathbb{C} : |\text{Im } z| < \omega\}$  the horizontal strip of height  $2\omega$ . We further define  $H^\infty(\text{Str}_\omega)$  to be the space of bounded holomorphic functions on  $\text{Str}_\omega$ , which is a Banach algebra equipped with the norm  $\|f\|_{\infty, \omega} = \sup_{\lambda \in \text{Str}_\omega} |f(\lambda)|$ . A densely defined operator  $B$  is called an operator of  $\omega$ -strip-type if  $\sigma(B) \subset \overline{\text{Str}_\omega}$  and for all  $\theta > \omega$  there is a  $C_\theta > 0$  such that  $\|\lambda(\lambda - B)^{-1}\| \leq C_\theta$  for all  $\lambda \in \overline{\text{Str}_\theta}^c$ . Similarly to the sectorial case, one defines  $f(B)$  for  $f \in H^\infty(\text{Str}_\theta)$  satisfying a decay for  $|\text{Re } \lambda| \rightarrow \infty$  by a Cauchy integral formula, and says that  $B$  has a bounded  $H^\infty(\text{Str}_\theta)$  calculus provided that  $\|f(B)\| \leq C\|f\|_{\infty, \theta}$ . In this case  $f \mapsto f(B)$  extends to a bounded homomorphism  $H^\infty(\text{Str}_\theta) \rightarrow B(X)$ . We refer to [13] and [28, Chapter 4] for details. We call  $B$  of 0-strip-type if  $B$  is  $\omega$ -strip-type for all  $\omega > 0$ .

There is an analogous statement to Lemma 4.1 which holds for a 0-strip-type operator  $B$  and  $\text{Str}_\omega$  in place of  $A$  and  $\Sigma_\omega$ , and  $\text{Hol}(\text{Str}_\omega) = \{f : \text{Str}_\omega \rightarrow \mathbb{C} : \exists n \in \mathbb{N} : (\rho \circ \exp)^n f \in H^\infty(\text{Str}_\omega)\}$ , where  $\rho(\lambda) = \lambda(1 + \lambda)^{-2}$ .

In fact, 0-strip-type operators and 0-sectorial operators with bounded  $H^\infty(\text{Str}_\omega)$  and bounded  $H^\infty(\Sigma_\omega)$  calculus are in one-one correspondence by the following lemma. For a proof we refer to [28, Proposition 5.3.3., Theorem 4.3.1 and Theorem 4.2.4, Lemma 3.5.1].

**Lemma 10.1.** Let  $B$  be an operator of 0-strip-type and assume that there exists a 0-sectorial operator  $A$  such that  $B = \log(A)$ . This is the case if  $B$  has a bounded  $H^\infty(\text{Str}_\omega)$  calculus for some  $\omega < \pi$ . Then for any  $f \in \bigcup_{0 < \omega < \pi} \text{Hol}(\text{Str}_\omega)$  one has

$$f(B) = (f \circ \log)(A).$$

Note that the logarithm belongs to  $\text{Hol}(\Sigma_\omega)$  for any  $\omega \in (0, \pi)$ . Conversely, if  $A$  is a 0-sectorial operator that has a bounded  $H^\infty(\Sigma_\omega)$  calculus for some  $\omega \in (0, \pi)$ , then  $B = \log(A)$  is an operator of 0-strip-type.

Let  $B$  be an operator of 0-strip-type,  $p \in [1, \infty)$  and  $\alpha > \frac{1}{p}$ . We say that  $B$  has a (bounded)  $W_p^\alpha$  calculus if there exists a constant  $C > 0$  such that

$$\|f(B)\| \leq C\|f\|_{W_p^\alpha} \quad (f \in \bigcap_{\omega > 0} H^\infty(\text{Str}_\omega) \cap W_p^\alpha).$$

In this case, by density of  $\bigcap_{\omega > 0} H^\infty(\text{Str}_\omega) \cap W_p^\alpha$  in  $W_p^\alpha$ , the definition of  $f(B)$  can be continuously extended to  $f \in W_p^\alpha$ .

Assume that  $B$  has a  $W_p^\alpha$  calculus for some  $\alpha > \frac{1}{p}$ . Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a function such that  $t \mapsto f(t)e^{ct}(1 + e^{ct})^{-2}$  belongs to  $W_p^\alpha$  for some  $c > 0$ . We define the operator  $f(B)$  to be the closure of

$$\begin{cases} D_B \subset X & \longrightarrow X \\ x & \longmapsto \sum_{n \in \mathbb{Z}} (\psi_n f)(B)x, \end{cases}$$

where  $D_B = \{x \in X : \exists N \in \mathbb{N} : \psi_n(B)x = 0 \text{ } (|n| \geq N)\}$  and  $(\psi_n)_{n \in \mathbb{Z}}$  is an equidistant partition of unity (see Lemma 3.4).

Then there is a version of Lemma 4.5, for which a proof can be found in [35, Proposition 4.25]. Let  $\widetilde{\mathcal{H}}_p^\alpha = \{f \in L_{\text{loc}}^p(\mathbb{R}) : \|f\|_{\widetilde{\mathcal{H}}_p^\alpha} = \sup_{n \in \mathbb{Z}} \|\psi_n f\|_{W_p^\alpha} < \infty\}$ .

Note that the Hörmander class  $\widetilde{\mathcal{H}}_p^\alpha$  is covered by the variant of Lemma 4.5. Let  $p \in (1, \infty)$ ,  $\alpha > \frac{1}{p}$  and let  $B$  be an operator of 0-strip-type. We say that  $B$  has a

(bounded)  $\widetilde{\mathcal{H}}_p^\alpha$  calculus if there exists a constant  $C > 0$  such that

$$\|f(B)\| \leq C\|f\|_{\widetilde{\mathcal{H}}_p^\alpha} \quad (f \in \bigcap_{\omega>0} H^\infty(\text{Str}_\omega) \cap \widetilde{\mathcal{H}}_p^\alpha).$$

The strip-type version of the main Theorem 6.1 reads as follows.

**Theorem 10.2.** Let  $B$  be 0-strip-type operator with  $H^\infty$  calculus on some Banach space with property  $(\alpha)$ . Denote  $U(t)$  the  $C_0$ -group generated by  $iB$ . For  $r \in (1, 2]$  and  $\alpha > \frac{1}{r}$ , consider the condition

$(C_r)_\alpha$   $B$  has an  $R$ -bounded  $\widetilde{\mathcal{H}}_r^\alpha$  calculus.

Furthermore, for  $\alpha \geq 0$ , we consider the conditions

(I) $_\alpha$  There exists  $C > 0$  such that for all  $t \in \mathbb{R}$ ,  $\|U(t)\| \leq C(1 + |t|)^\alpha$ .

(II) $_\alpha$  The set  $\{\langle t \rangle^{-\alpha} U(t) : t \in \mathbb{R}\}$  is (semi-)  $R$ -bounded.

Then the following hold.

- (a) Let  $r \in (1, 2]$  such that  $\frac{1}{r} > \frac{1}{\text{type } X} - \frac{1}{\text{cotype } X}$  and  $\beta > \alpha + \frac{1}{r}$ . Then (I) $_\alpha$  implies  $(C_r)_\beta$ .
- (b) Consider  $\alpha, \beta \geq 0$  with  $\beta > \alpha + \frac{1}{2}$ . Then (II) $_\alpha$  implies  $(C_2)_\beta$ .

*Proof.* Considering the 0-sectorial operator  $A = e^B$ , the Theorem follows at once from the sectorial Theorem 6.1 together with Lemma 10.1.  $\square$

## REFERENCES

- [1] G. Alexopoulos. Spectral multipliers on Lie groups of polynomial growth. *Proc. Am. Math. Soc.* 120(3):973–979, 1994. 1, 26
- [2] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Grundlehren der mathematischen Wissenschaften, 223. Berlin etc.: Springer, 1976. 13, 30, 31
- [3] S. Blunck. A Hörmander-type spectral multiplier theorem for operators without heat kernel. *Ann. Sc. Norm. Sup. Pisa (5)* 2(3):449–459, 2003. 1, 26, 27
- [4] S. Blunck. Generalized Gaussian estimates and Riesz means of Schrödinger groups. *J. Aust. Math. Soc.* 82(2):149–162, 2007. 2, 27
- [5] S. Blunck and P. Kunstmann. Generalized Gaussian estimates and the Legendre transform. *J. Oper. Theory* 53(2):351–365, 2005. 2, 27
- [6] S. Blunck and P. C. Kunstmann. Calderón-Zygmund theory for non-integral operators and the  $H^\infty$  functional calculus. *Rev. Mat. Iberoam.* 19(3):919–942, 2003. 26
- [7] A. Bonami and J.-L. Clerc. Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques. *Trans. Amer. Math. Soc.* 183:223–263, 1973. 34
- [8] J. Bourgain. Vector valued singular integrals and the  $H^1$ –BMO duality. *Probability theory and harmonic analysis (Cleveland, Ohio, 1983) Monogr. Textbooks Pure Appl. Math., Vol. 98*, p.1–19 Dekker, New York, 1986. 4
- [9] K. Boyadzhiev and R. deLaubenfels. Boundary values of holomorphic semigroups. *Proc. Am. Math. Soc.* 118(1):113–118, 1993. 23
- [10] P. Chen, E. M. Ouhabaz, A. Sikora and L. Yan. Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. Preprint online at <http://arxiv.org/abs/1202.4052>. 1, 12, 25, 27
- [11] M. Christ.  $L^p$  bounds for spectral multipliers on nilpotent groups. *Trans. Am. Math. Soc.* 328(1):73–81, 1991. 1, 26
- [12] M. G. Cowling. Harmonic analysis on semigroups. *Ann. Math.* 117:267–283, 1983. 19, 34
- [13] M. Cowling, I. Doust, A. McIntosh and A. Yagi. Banach space operators with a bounded  $H^\infty$  functional calculus. *J. Aust. Math. Soc., Ser. A* 60(1):51–89, 1996. 1, 9, 31, 36
- [14] M. Cwikel and S. Reisner. Interpolation of uniformly convex Banach spaces. *Proc. Amer. Math. Soc.* 84(4):555–559, 1982. 28, 29

- [15] E. Davies. *Heat kernels and spectral theory*. Cambridge Tracts in Mathematics, 92. Cambridge etc.: Cambridge University Press, 1989. 26
- [16] R. deLaubenfels and Y. Lei. Regularized functional calculi, semigroups, and cosine functions for pseudodifferential operators. *Abstr. Appl. Anal.* (2)1-2:121–136, 1997. 23
- [17] L. Deleaval and C. Kriegler. Spectral multipliers with values in UMD lattices. in preparation. 28
- [18] J. Diestel, H. Jarchow and A. Tonge. *Absolutely summing operators*. Cambridge Studies in Advanced Mathematics, 43. Cambridge: Cambridge Univ. Press, 1995. 4, 5
- [19] M. Duelli and L. Weis. Spectral projections, Riesz transforms and  $H^\infty$ -calculus for bisectorial operators. *Nonlinear elliptic and parabolic problems, Progress in Nonlinear Differential Equations Appl.* 64:99–111, Birkhäuser, Basel, 2005. 35
- [20] X. T. Duong. From the  $L^1$  norms of the complex heat kernels to a Hörmander multiplier theorem for sub-Laplacians on nilpotent Lie groups. *Pac. J. Math.* 173(2):413–424, 1996. 1, 22, 26
- [21] X. T. Duong, E. M. Ouhabaz and A. Sikora. Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.* 196(2):443–485, 2002. 1, 9, 26, 27
- [22] X. T. Duong, E. M. Ouhabaz and L. Yan. Weighted norm inequalities, Gaussian bounds and sharp spectral multipliers. *J. Funct. Anal.* 260(4):1106–1131, 2011. 1
- [23] S. Fackler. The Kalton-Lancien theorem revisited: maximal regularity does not extrapolate. *J. Funct. Anal.* 266(1):121–138, 2014. 7
- [24] G. Folland and E. Stein. *Hardy spaces on homogeneous groups*. Mathematical Notes, 28. Princeton, NJ: Princeton University Press, University of Tokyo Press, 1982. 26
- [25] O. van Gaans. On  $R$ -boundedness of unions of sets of operators. *PDE and funct. anal., Oper. Theory Adv. Appl.* 168, 97–111, 2006. 7
- [26] J. E. Galé and P. J. Miana.  $H^\infty$  functional calculus and Mikhlin-type multiplier conditions. *Can. J. Math.* 60(5):1010–1027, 2008. 15, 34
- [27] J. E. Galé and T. Pytlik. Functional calculus for infinitesimal generators of holomorphic semigroups. *J. Funct. Anal.* 150(2):307–355, 1997. 3, 34
- [28] M. Haase. *The functional calculus for sectorial operators*. Operator Theory: Advances and Applications, 169. Basel: Birkhäuser, 2006. 12, 36
- [29] B. H. Haak and P. C. Kunstmann. Weighted admissibility and wellposedness of linear systems in Banach spaces. *SIAM J. Control Optim.* 45(6):2094–2118, 2007. 18, 20
- [30] T. Hytönen and M. Veraar.  $R$ -boundedness of smooth operator-valued functions. *Integral Equations Oper. Theory* 63(3):373–402, 2009. 5, 6
- [31] L. Hörmander. Estimates for translation invariant operators in  $L^p$  spaces. *Acta Math.* 104:93–140, 1960. 1
- [32] L. Hörmander. *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. 2nd ed.* Grundlehren der Mathematischen Wissenschaften, 256. Berlin etc.: Springer, 1990. 9
- [33] N. Kalton. A remark on sectorial operators with an  $H^\infty$  calculus. *Contemp. Math.* 321, 2003. 28, 29, 30
- [34] N. Kalton and L. Weis. The  $H^\infty$ -calculus and square function estimates. Preprint, online at <https://arxiv.org/pdf/1411.0472v2.pdf> 20
- [35] C. Kriegler. Spectral multipliers,  $R$ -bounded homomorphisms, and analytic diffusion semigroups. PhD-thesis, online at <http://digbib.ubka.uni-karlsruhe.de/volltexte/1000015866> 9, 10, 15, 33, 36
- [36] C. Kriegler. Hörmander Type Functional Calculus and Square Function Estimates. *J. Oper. Theory* 71(1):223–257, 2014. 4, 25
- [37] C. Kriegler. Hörmander functional calculus for Poisson estimates. *Int. Equ. Oper. Theory* 80(3):379–413, 2014. 25
- [38] C. Kriegler and L. Weis. Paley-Littlewood Decomposition for Sectorial Operators and Interpolation Spaces. *Math. Nachr.* 289(11-12):1488–1525, 2016. 4, 13
- [39] C. Kriegler and L. Weis. Spectral multiplier theorems and averaged  $R$ -boundedness. To appear in Semigroup Forum. Preprint, online at <http://arxiv.org/abs/1407.0194> 4, 13, 14, 20, 33

- [40] C. Kriegler and C. Le Merdy. Tensor extension properties of  $C(K)$ -representations and applications to unconditionality. *J. Aust. Math. Soc.* 88(2):205–230, 2010. 10
- [41] P. C. Kunstmann. On maximal regularity of type  $L^p - L^q$  under minimal assumptions for elliptic non-divergence operators. *J. Funct. Anal.* 255(10):2732–2759, 2008. 27
- [42] P. C. Kunstmann and M. Uhl. Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces. *J. Operator Theory* 73(1):27–69, 2015. 27
- [43] P. C. Kunstmann and L. Weis. Maximal  $L_p$ -regularity for parabolic equations, Fourier multiplier theorems and  $H^\infty$ -functional calculus. *Functional analytic methods for evolution equations. Based on lectures given at the autumn school on evolution equations and semigroups, Levico Terme, Trento, Italy, October 28–November 2, 2001.* Berlin: Springer, Lect. Notes Math. 1855, 65–311, 2004. 2, 4, 5, 7, 8, 9, 12, 17, 20, 21, 23, 25, 35
- [44] C. Le Merdy.  $H^\infty$ -functional calculus and applications to maximal regularity. *Semi-groupes d'opérateurs et calcul fonctionnel. Ecole d'été, Besançon, France, Juin 1998.* Besançon: Université de Franche-Comté et CNRS, Equipe de Mathématiques. Publ. Math. UFR Sci. Tech. Besançon 16, 41–77, 1998. 31, 32
- [45] C. Le Merdy. On square functions associated to sectorial operators. *Bull. Soc. Math. France* 132(1):137–156, 2004. 18, 20
- [46] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces I. Sequence spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 92. Berlin etc.: Springer, 1977. 31
- [47] G. Mauceri and S. Meda. Vector-valued multipliers on stratified groups. *Rev. Mat. Iberoam.* 6(3-4):141–154, 1990. 26
- [48] S. Meda. A general multiplier theorem. *Proc. Am. Math. Soc.* 110(3):639–646, 1990. 1, 19
- [49] D. Müller. Functional calculus of Lie groups and wave propagation. *Doc. Math., J. DMV Extra Vol. ICM Berlin* 679-689, 1998. 1, 23
- [50] D. Müller and E. Stein. On spectral multipliers for Heisenberg and related groups. *J. Math. Pures Appl. (9)* 73(4):413–440, 1994. 1, 26
- [51] J. van Neerven.  $\gamma$ -radonifying operators: a survey. *Proc. Centre Math. Appl. Austral. Nat. Univ.* 44:1–61, 2010. 18, 20
- [52] E. M. Ouhabaz. *Analysis of heat equations on domains.* London Mathematical Society Monographs, 31. Princeton, NJ: Princeton University Press, 2005. 1, 23, 26
- [53] E. M. Ouhabaz. Sharp Gaussian bounds and  $L^p$ -growth of semigroups associated with elliptic and Schrödinger operators. *Proc. Am. Math. Soc.* 134(12):3567–3575, 2006. 26
- [54] G. Pisier. Some results on Banach spaces without local unconditional structure. *Compositio Math.* 37(1):3–19, 1978. 30
- [55] E. M. Stein. Interpolation of linear operators. *Trans. Am. Math. Soc.* 83:482–492, 1956. 11
- [56] K. Stempak. Multipliers for eigenfunction expansions of some Schrödinger operators. *Proc. Am. Math. Soc.* 93(3):477–482, 1985. 34
- [57] P. A. Tomas. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.* 81, 477–478, 1975. 25
- [58] H. Triebel. *Interpolation theory, function spaces, differential operators.* North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York, 1978. 528 pp. 30
- [59] M. Uhl. Spectral multiplier theorems of Hörmander type via generalized Gaussian estimates. PhD-thesis, online at <http://digbib.ubka.uni-karlsruhe.de/volltexte/1000025107> 27
- [60] N. Varopoulos. Analysis on Lie groups. *J. Funct. Anal.* 76(2):346–410, 1988. 26
- [61] M. Veraar and L. Weis. On semi- $R$ -boundedness and its applications. *J. Math. Anal. Appl.* 363:431–443, 2010. 5, 19

CHRISTOPH KRIEGLER, LABORATOIRE DE MATHÉMATIQUES (CNRS UMR 6620), UNIVERSITÉ BLAISE-PASCAL (CLERMONT-FERRAND 2), CAMPUS UNIVERSITAIRE DES CÉZEAUX, 3, PLACE VASARELY, TSA 60026, CS 60026, 63 178 AUBIÈRE CEDEX, FRANCE  
*E-mail address:* christoph.kriegler@math.univ-bpclermont.fr

LUTZ WEIS, KARLSRUHER INSTITUT FÜR TECHNOLOGIE, FAKULTÄT FÜR MATHEMATIK, INSTITUT FÜR ANALYSIS, ENGLERSTRASSE 2, 76131 KARLSRUHE, GERMANY  
*E-mail address:* lutz.weis@kit.edu